SectionBuilder User's Manual

Definition of material properties

Contents

1 Material stiffness properties

The material stiffness properties are defined in this section. Three types of linear elastic materials are considered here, *isotropic*, *orthotropic*, or *transversely isotropic* materials [1, 2, 3].

Linear elastic materials fall into three categories: isotropic, orthotropic, or transversely isotropic. Orthotropic materials possess two orthogonal planes of material property symmetry, implying the existence of a third as illustrated in fig. 1. Transversely isotropic materials feature one plane of material isotropy, *i.e.* properties are identical in all directions in this plane. Typically, advanced composite materials are transversely isotropic. Finally, isotropic material have identical properties in all directions. In each case, a **material basis**, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$, is defined that reflects the possible existence of various planes of symmetry and/or orthotropy, as illustrated in fig. 1.

The constitutive laws for linear elastic materials will be cast as linear relationships between stress and strain components. The array of *strain components*, resolved in the material basis, is defined as

$$\underline{\epsilon}^{+T} = \lfloor \epsilon_1^+, \epsilon_2^+, \epsilon_3^+, \gamma_{23}^+, \gamma_{13}^+, \gamma_{12}^+ \rfloor, \tag{1}$$

where ϵ_1^+, ϵ_2^+ and ϵ_3^+ are the axial strains components along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively. The corresponding engineering shear strains components are γ_{23}^+, γ_{13}^+ and γ_{12}^+. Notation $(\cdot)^+$ indicates tensor components resolved in basis $\mathcal{E}^+$. The array of *stress components*, resolved in the material basis, is

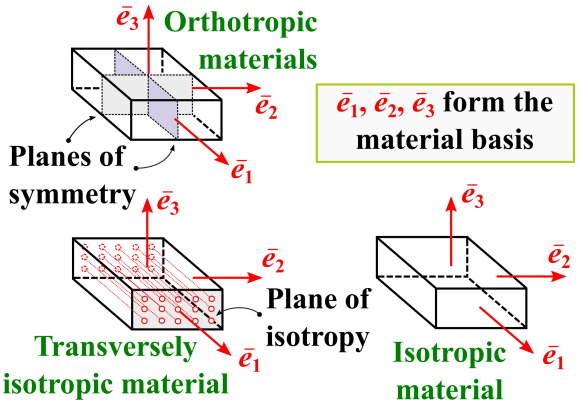

Figure 1: Orthotropic, transversely isotropic and isotropic materials.

$$\underline{\sigma}^{+T} = \lfloor \sigma_1^+, \sigma_2^+, \sigma_3^+, \tau_{23}^+, \tau_{13}^+, \tau_{12}^+ \rfloor, \tag{2}$$

where σ_1^+, σ_2^+ and σ_3^+ are the axial stress components along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively; the corresponding shearing stresses are τ_{23}^+, τ_{13}^+ and τ_{12}^+.

A linear elastic material is characterized by the following constitutive laws

$$\underline{\sigma}^+ = \underline{\underline{C}}^+ \underline{\epsilon}^+, \tag{3a}$$

$$\underline{\epsilon}^+ = \underline{\underline{S}}^+ \underline{\sigma}^+, \tag{3b}$$

where $\underline{\underline{C}}^+$ is a 6×6 stiffness matrix and $\underline{\underline{S}}^+$ a 6×6 compliance matrix. Clearly, these two matrices are the inverse of each other, *i.e.*

$$\underline{\underline{C}}^{+-1} = \underline{\underline{S}}^+. \tag{4}$$

The strain energy, A, stored in a differential element of the material is

$$A = \frac{1}{2}\, \underline{\epsilon}^{+T} \underline{\sigma}^+ = \frac{1}{2}\, \underline{\epsilon}^{+T} \underline{\underline{C}}^+ \underline{\epsilon}^+ = \frac{1}{2}\, \underline{\sigma}^T \underline{\underline{S}}^+ \underline{\sigma}^+. \tag{5}$$

The stored strain energy is a positive quantity for whatever deformation or stress state the material is subjected to. This implies that both stiffness and compliance matrices are symmetric and definite positive.

1.1 Orthotropic materials

An *orthotropic material* has at least two orthogonal planes of material property symmetry. For these materials, the compliance matrix takes the following form

$$\underline{\underline{\mathcal{S}}}^{+} = \begin{bmatrix} 1/E_1^+ & -\nu_{21}^+/E_2^+ & -\nu_{31}^+/E_3^+ & 0 & 0 & 0 \\ -\nu_{12}^+/E_1^+ & 1/E_2^+ & -\nu_{32}^+/E_3^+ & 0 & 0 & 0 \\ -\nu_{13}^+/E_1^+ & -\nu_{23}^+/E_2^+ & 1/E_3^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{13}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12}^+ \end{bmatrix}. \tag{6}$$

The stiffness coefficients appearing in this compliance matrix are three distinct Young's moduli, E_1^+, E_2^+, and E_3^+, three Poisson's ratios, ν_{12}^+, ν_{13}^+, and ν_{23}^+, and three shearing moduli, G_{12}^+, G_{13}^+, and G_{23}^+. Because this matrix is symmetric, the following relationships hold

$$\nu_{23}^+/E_2^+ = \nu_{32}^+/E_3^+, \quad \nu_{31}^+/E_3^+ = \nu_{13}^+/E_1^+, \quad \nu_{21}^+/E_2^+ = \nu_{12}^+/E_1^+. \tag{7}$$

In view of this relationship, the material stiffness properties are characterized by three distinct **Young's moduli**: E_1^+, E_2^+ and E_3^+ along unit vectors $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively; three **Poisson's ratios**: ν_{12}^+, ν_{13}^+ and ν_{23}^+; and three **shearing moduli**: G_{12}^+, G_{13}^+ and G_{23}^+. Thus, for an *orthotropic material*, the following *nine* properties are required: *(1)* **Young's moduli:** E_1^+, E_2^+ and E_3^+; *(2)* **Shear moduli:** G_{12}^+, G_{13}^+ and G_{23}^+; *(3)* **Poisson's ratios:** ν_{12}^+, ν_{13}^+ and ν_{23}^+.

1.2 Transversely isotropic materials

A *transversely isotropic material* is an orthotropic material, additionally presenting a plane of material property isotropy. As illustrated in fig. 1, plane $(\bar{e}_2, \bar{e}_3)$ will be selected as the plane of isotropy. In this case, $E_3^+ = E_2^+$, $G_{13}^+ = G_{12}^+$ and $\nu_{13}^+ = \nu_{12}^+$: in view of the isotropy in the $(\bar{e}_2, \bar{e}_3)$ plane, the subscripts $(.)_2$ and $(.)_3$ can be interchanged. Furthermore, the isotropy of plane $(\bar{e}_2, \bar{e}_3)$ implies $G_{23}^+ = E_2^+/[2(1+\nu_{23}^+)]$. For these materials, the compliance matrix takes the following form

$$\underline{\underline{\mathcal{S}}}^{+} = \begin{bmatrix} 1/E_1^+ & -\nu_{12}^+/E_1^+ & -\nu_{12}^+/E_1^+ & 0 & 0 & 0 \\ -\nu_{12}^+/E_1^+ & 1/E_2^+ & -\nu_{23}^+/E_2^+ & 0 & 0 & 0 \\ -\nu_{12}^+/E_1^+ & -\nu_{23}^+/E_2^+ & 1/E_2^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu_{23}^+)/E_2^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{12}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12}^+ \end{bmatrix}. \tag{8}$$

For transversely isotropic elastic materials, the following *five* properties are required: *(1)* **Young's moduli:** E_1^+ and E_2^+; *(2)* **Shear moduli:** G_{12}^+, *(3)* **Poisson's ratios:** ν_{12}^+ and ν_{23}^+.

1.3 Isotropic materials

An *isotropic material* is a material that presents identical properties in all directions. In this case, the isotropy of the material implies $E_1^+ = E_2^+ = E_3^+ = E$, $\nu_{12}^+ = \nu_{13}^+ = \nu_{23}^+ = \nu$, and $G_{12}^+ = G_{13}^+ = G_{23}^+ = E/[2(1+\nu)]$. For these materials, the compliance matrix takes the following form

$$\underline{\underline{\mathcal{S}}}^{+} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu)/E & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu)/E & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)/E \end{bmatrix}. \tag{9}$$

For isotropic materials, the following *two* properties are required: *(1)* **Young's modulus:** E, *(2)* **Poisson's ratios:** ν.

2 Allowable stresses

In general, the strength properties of a material can be defined by nine different stress values. *(1) Allowable Tensile Stress*: σ_1^{aT}, σ_2^{aT} and σ_3^{aT}, along the material axes $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively.*(2) Allowable Compressive Stress*: σ_1^{aC}, σ_2^{aC} and σ_3^{aC}, along the material axes $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$, respectively. *(3)Allowable Shear Stress*: τ_{12}^{a}, τ_{13}^{a} and τ_{23}^{a}.

Table 1: Notation for the various allowable stress components in the material axis system

	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$
Tensile stress	σ_1^{aT}	σ_2^{aT}	σ_3^{aT}
Compressive stress	σ_1^{aC}	σ_2^{aC}	σ_3^{aC}
Shear stress	τ_{12}^{a}	τ_{13}^{a}	τ_{23}^{a}

For different types of materials, the required number of strength properties are different.

1. For an *orthotropic material*, all *nine* strength properties are required: *(1)* **Tension strength:** σ_1^{aT}, σ_2^{aT} and σ_3^{aT}; *(2)* **Compression strength:** σ_1^{aC},σ_2^{aC} and σ_3^{aC}; *(3)* **Shear strength:** τ_{12}^{a}, τ_{13}^{a} and τ_{23}^{a}.

2. For a *transversely isotropic material*, the following *five* properties are required: *(1)* **Tension strength:** σ_1^{aT} and σ_2^{aT}; *(2)* **Compression strength:** σ_1^{aC} and σ_2^{aC}; *(3)* **Shear strength:** τ_{12}^{a}. In this case, $\sigma_3^{\mathrm{aT}} = \sigma_2^{\mathrm{aT}}$, $\sigma_3^{\mathrm{aC}} = \sigma_2^{\mathrm{aC}}$ and $\tau_{13}^{\mathrm{a}} = \tau_{12}^{\mathrm{a}}$: in view of the isotropy in the ($\bar{e}_2$, $\bar{e}_3$) plane, the subscripts $(.)_2$ and $(.)_3$ can be interchanged. Furthermore, the isotropy of plane ($\bar{e}_2$, $\bar{e}_3$) implies $\tau_{23}^{\mathrm{a}} = \sigma_2^{\mathrm{aT}}/\sqrt{3}$.

3. For an *isotropic material*, a *single* property is required: **Tension strength:** σ^{a}. In this case, $\sigma_1^{\mathrm{aT}} = \sigma_2^{\mathrm{aT}} = \sigma_3^{\mathrm{aT}} = \sigma^{\mathrm{a}}$, $\sigma_1^{\mathrm{aC}} = \sigma_2^{\mathrm{aC}} = \sigma_3^{\mathrm{aC}} = \sigma^{\mathrm{a}}$ and $\tau_{12}^{\mathrm{a}} = \tau_{13}^{\mathrm{a}} = \tau_{23}^{\mathrm{a}} = \sigma^{\mathrm{a}}/\sqrt{3}$.

3 Failure criteria

Different failure criteria are used for different material types. For instance, Von Mises' criterion, see section 3.2, was developed for homogeneous, isotropic materials. When dealing with transversely isotropic materials, several criteria have been developed, such as Hoffmann's criterion, see section 3.3, Tsai-Wu's criterion, see section 3.4, or the maximum stress and strain criteria, see sections 3.5 and 3.6, respectively.

3.1 The reserve factor

The concept of reserve factor is often used in stress computations. The reserve factor, R, is defined as the factor by which the applied stress can be multiplied to reach failure, *i.e.*,

$$\sigma_{\mathrm{fail}} = R\sigma_{\mathrm{appl}}. \tag{10}$$

From this definition it follows that:

- $R = 1$ means that the applied stresses causes failure;

- $R > 1$ means that the applied stresses level is safe, *i.e.*, it is below the failure level. A reserve factor of two means that the applied stresses can be doubled before failure occurs;
- $R < 1$ means that the applied stresses is above the failure stress.

3.2 Von Mises' criterion

Von Mises' criterion was developed for *homogeneous, isotropic materials*. The **principal stresses**, σ_1, σ_2, and σ_3, are written in a non-dimensional form as

$$s_1 = \frac{\sigma_1}{\sigma^{\mathrm{aT}}}, \quad s_2 = \frac{\sigma_2}{\sigma^{\mathrm{aT}}}, \quad s_3 = \frac{\sigma_3}{\sigma^{\mathrm{aT}}}, \tag{11}$$

where σ^{aT} the allowable stress in tension defined in section 2. Because the material is isotropic, its strength is identical in all directions; furthermore, its compressive and tensile strengths are assumed to be identical. Von Mises' criterion postulates that at failure, the following equation is satisfied

$$\frac{1}{2}\left[(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2\right] = 1. \tag{12}$$

If (s_1, s_2, s_3) are the principal stress components applied to the material, the definition of the reserve factor, eq. (10), implies that principal stress components (Rs_1, Rs_2, Rs_3) cause failure of the material and hence, these stress components should satisfy Von Mises' criterion, eq. (12), leading to

$$R_{1,2} = \pm\sqrt{\frac{2}{(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2}}. \tag{13}$$

One reserve factor is positive, the other is negative. The positive reserve factor gives the stress level at failure and the negative reserve factor gives the stress level at failure when the sign of the applied stresses is reversed. Because the strengths of the material are assumed to be identical in tension and compression, the two reserve factors has the same absolute value.

3.3 Hoffmann's criterion

Hoffmann's criterion was developed for *transversely isotropic material*. The following non-dimensional stress components in the material system are defined first

$$s_1 = \frac{\sigma_1}{\sqrt{\sigma_1^{\mathrm{aT}}\sigma_1^{\mathrm{aC}}}}, \quad s_2 = \frac{\sigma_2}{\sqrt{\sigma_2^{\mathrm{aT}}\sigma_2^{\mathrm{aC}}}}, \quad s_6 = \frac{\tau_{12}}{\tau_{12}^{\mathrm{a}}}, \tag{14}$$

where allowable stress components σ_1^{aT}, σ_1^{aC}, σ_2^{aT}, σ_2^{aC}, and τ_{12}^{a} are listed in table 1. Hoffmann's criterion postulates that at failure, the following equation is satisfied

$$s_1^2 - F_{12}s_1s_2 + s_2^2 + s_6^2 + F_1s_1 + F_2s_2 = 1, \tag{15}$$

where coefficients F_1, F_2, and F_{12} are defined as

$$F_1 = \frac{\sigma_1^{\mathrm{aC}} - \sigma_1^{\mathrm{aT}}}{\sqrt{\sigma_1^{\mathrm{aT}}\sigma_1^{\mathrm{aC}}}}, \quad F_2 = \frac{\sigma_2^{\mathrm{aC}} - \sigma_2^{\mathrm{aT}}}{\sqrt{\sigma_2^{\mathrm{aT}}\sigma_2^{\mathrm{aC}}}}, \quad F_{12} = \sqrt{\frac{\sigma_2^{\mathrm{aT}}\sigma_2^{\mathrm{aC}}}{\sigma_1^{\mathrm{aT}}\sigma_1^{\mathrm{aC}}}}. \tag{16}$$

If (s_1, s_2, s_6) are the stress components applied to the material, the definition of the reserve factor, eq. (10), implies that stress components (Rs_1, Rs_2, Rs_6) cause failure of the material and hence, these stress components should satisfy Hoffmann's criterion, eq. (15), leading to

$$R^2\left[s_1^2 - F_{12}s_1s_2 + s_2^2 + s_6^2\right] + R\left[F_1s_1 + F_2s_2\right] - 1 = 0.$$

Let coefficients a and b group the terms that are quadratic and linear in the stress components, respectively, *i.e.*,

$$a = s_1^2 - F_{12} s_1 s_2 + s_2^2 + s_6^2, \tag{17a}$$
$$b = F_1 s_1 + F_2 s_2. \tag{17b}$$

Hoffmann's criterion now reduces to a simple quadratic equation, $aR^2 + bR - 1 = 0$, and its two roots are denoted R_1 and R_2, respectively. One reserve factor is positive $R_1 \geq 0$, the other is negative, $R_2 \leq 0$. The positive reserve factor gives the stress level at failure and the negative reserve factor gives the stress level at failure when the sign of the applied stresses is reversed. Because the strengths of the material in tension and compression differ, the two reserve factors differ, in general, $|R_1| \neq |R_2|$.

The quadratic equation must be solved with care. Let ϵ denote a small positive number. Because coefficient a is positive, $a \leq \epsilon$ implies the vanishing of all the stress components, $a \approx 0$ and $b \approx 0$, and hence, $R_1 = 1/\epsilon$ and $R_1 = -1/\epsilon$. If a does not vanish, the solution of the quadratic equation is written as

$$q = -\frac{1}{2}\left[b + \text{sign}(b)\sqrt{b^2 + 4a}\right], \quad R_1 = \frac{q}{a}, \quad R_2 = -\frac{1}{q},$$

to avoid division by zero [4]. Table 2 summarizes the solution process.

Table 2: Solutions of the quadratic equations for the reserve factors

	$a \leq \epsilon$	$a > \epsilon, \quad b \geq 0$	$a > \epsilon, \quad b \leq 0$
q	0	$-(b/2) - \sqrt{(b/2)^2 + a} < 0$	$-(b/2) + \sqrt{(b/2)^2 + a} > 0$
R_1	$1/\epsilon$	$-1/q$	q/a
R_2	$-1/\epsilon$	q/a	$-1/q$

3.4 Tsai-Wu's criterion

Tsai-Wu's criterion criterion has been developed for *transversely isotropic material*. The non-dimensional stress components defined by eq. (14) are introduced and Tsai-Wu's criterion postulates that at failure, the following equation is satisfied

$$s_1^2 - s_1 s_2 + s_2^2 + s_6^2 + F_1 s_1 + F_2 s_2 = 1, \tag{18}$$

where coefficients F_1 and F_2 are defined as

$$F_1 = \frac{\sigma_1^{\text{aC}} - \sigma_1^{\text{aT}}}{\sqrt{\sigma_1^{\text{aT}} \sigma_1^{\text{aC}}}}, \tag{19a}$$
$$F_2 = \frac{\sigma_2^{\text{aC}} - \sigma_2^{\text{aT}}}{\sqrt{\sigma_2^{\text{aT}} \sigma_2^{\text{aC}}}}. \tag{19b}$$

The reserve factors can be evaluated as the two root of a quadratic equation, $aR^2 + bR - 1 = 0$, where coefficients a and b are defined as

$$a = s_1^2 - s_1 s_2 + s_2^2 + s_6^2, \tag{20a}$$
$$b = F_1 s_1 + F_2 s_2. \tag{20b}$$

The solution of this quadratic equation is detailed in section 3.3, see table 2.

3.5 Maximum stress criterion

This maximum stress criterion was developed *orthotropic materials*. If the material is stressed below the failure limit, the following inequalities must be satisfied,

$$\sigma_i = \begin{cases} \sigma_i^{\mathrm{aT}} & \text{if } \sigma_i > 0, \\ \sigma_i^{\mathrm{aC}} & \text{if } \sigma_i < 0, \end{cases} \tag{21a}$$

$$\tau_{ij} = \tau_{ij}^{\mathrm{a}}, \tag{21b}$$

where σ_i is the stress component acting **along material axis** $\bar{e}_i$ and τ_{ij} the corresponding shear stress components. The allowable tensile and compressive stress components along unit vector $\bar{e}_i$ are denoted σ_i^{aT} and σ_i^{aC}, respectively. The allowable shear stress component in plane $(\bar{e}_i, \bar{e}_j)$ is denoted τ_{ij}^{a}. All these quantities are defined in section 2.

3.6 Maximum strain criterion

This criterion is used for *transversely isotropic material only*. At failure, one of the following equations is satisfied.

$$\epsilon_1 = \begin{cases} \sigma_1^{\mathrm{aT}}/E_1^+ & \text{if } \epsilon_1 > 0, \\ \sigma_1^{\mathrm{aC}}/E_1^+ & \text{if } \epsilon_1 < 0, \end{cases} \tag{22a}$$

$$\epsilon_2 = \begin{cases} \sigma_2^{\mathrm{aT}}/E_2^+ & \text{if } \epsilon_2 > 0, \\ \sigma_2^{\mathrm{aC}}/E_2^+ & \text{if } \epsilon_2 < 0, \end{cases}, \tag{22b}$$

$$\gamma_{12} = \tau_{12}^{\mathrm{a}}/G_{12}^+, \tag{22c}$$

where ϵ_1 and ϵ_2 are the **strains along the material axes** $\bar{e}_1$ and $\bar{e}_2$, respectively, and γ_{12} the corresponding shear strain. The allowable tensile and compressive stresses along unit vector $\bar{e}_1$ are denoted σ_1^{aT} and σ_1^{aC}, respectively. Similarly, the allowable tensile and compressive stresses along unit vector $\bar{e}_2$ are denoted σ_2^{aT} and σ_2^{aC}, respectively. Finally, the allowable shear stresses in plane $(\bar{e}_1, \bar{e}_2)$ is denoted τ_{12}^{a}. All these quantities are defined in section 2.

4 Rotation of material properties

In section 1, material stiffness properties were defined in the material basis, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$. When using these properties to evaluate the stiffness matrix of solid elements, it be be necessary to rotate these properties to the reference basis, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$. The definition of the relative orientation of the material basis with respect to the reference basis is performed based on *layer orientation angles* and two options, the "local basis option" and the "global basis option," are possible to define this relative orientation. These two options are described in sections 4.1 and 4.2, respectively.

4.1 The local basis option

When using the local basis option, three bases are involved in the determination of the orientation of the material basis: the *global reference basis*, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$, the *solid local basis*, $\mathcal{U} = (\bar{\imath}_1, \bar{u}, \bar{v})$, and the *material basis*, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$. The orientation of the solid local basis, $\mathcal{U}$, and two orientation angles, β and γ, are used to define the orientation of the material basis, $\mathcal{E}^+$.

A sequence of *three planar rotations* brings the reference basis, $\mathcal{I}$, to the material basis, $\mathcal{E}^+$, as illustrated in fig. 2.

1. The *first planar rotation* is of magnitude α about unit vector $\bar{\imath}_1$ and brings the reference basis, $\mathcal{I}$, to the solid local basis, $\mathcal{U}$. Angle α is determined by the geometry of the finite element.

2. The *second planar rotation* is of magnitude β about unit vector $\bar{\imath}_1$ and brings the solid local basis, $\mathcal{U}$, to basis $\mathcal{B} = (\bar{\imath}_1, \bar{b}_2, \bar{b}_3)$. Angle β is a user input. Because these first two planar rotations take place about the same unit vector, $\bar{\imath}_1$, they can be combined into a single planar rotation of magnitude $(\alpha + \beta)$ about unit vector $\bar{\imath}_1$.

3. The *third planar rotation* is of magnitude γ about unit vector $\bar{b}_3$ and brings basis $\mathcal{B}$, to the material basis, $\mathcal{E}^+$. Angle γ is a user input.

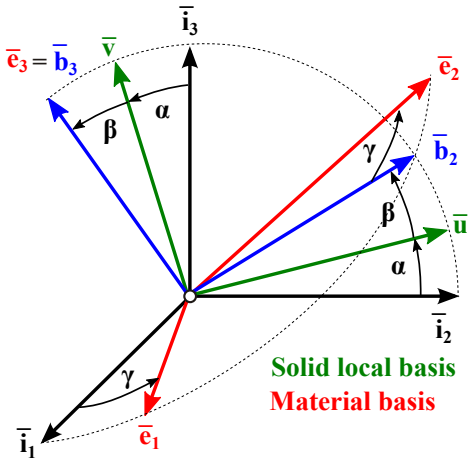

Figure 2: Orientation of the material basis using the **Local Axes** option.

With the help of fig. 3, it is verified readily that the rotation tensor, $\underline{\underline{R}}$ that brings basis $\mathcal{I}$ to basis $\mathcal{E}^+$ is

$$\underline{\underline{R}} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma\cos(\alpha+\beta) & \cos\gamma\cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin\gamma\sin(\alpha+\beta) & \cos\gamma\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \tag{23}$$

Note that positive angles β and γ correspond to positive rotations about axes $\bar{\imath}_1$ and $\bar{b}_3$, respectively, following the right-hand rule. If the layer is a transversely isotropic material such as a unidirectional layer of composite, angle $\beta = 0$ and angle γ corresponds to the fiber orientation angle.

4.2 The global basis option

When using the global basis option, two bases are involved in the determination of the orientation of the material basis: the *global reference basis*, $\mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)$ and the *material basis*, $\mathcal{E}^+ = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$. The two orientation angles, β and γ, are used to define the orientation of the material basis, $\mathcal{E}^+$.

A sequence of *two planar rotations* brings the reference basis, $\mathcal{I}$, to the material basis, $\mathcal{E}^+$, as illustrated in fig. 3.

1. The *first planar rotation* is of magnitude β about unit vector $\bar{\imath}_1$ and brings the global reference basis, $\mathcal{I}$, to frame $\mathcal{B} = (\bar{\imath}_1, \bar{b}_2, \bar{b}_3)$. Angle β is a user input.

2. The *second planar rotation* is of magnitude γ about unit vector $\bar{b}_3$ and brings basis $\mathcal{B}$, to the material basis, $\mathcal{E}^+$.

With the help of fig. 3, it is verified readily that the rotation tensor, $\underline{\underline{R}}$ that brings basis $\mathcal{I}$ to basis $\mathcal{E}^+$ is

$$\underline{\underline{R}} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma\cos\beta & \cos\gamma\cos\beta & -\sin\beta \\ \sin\gamma\sin\beta & \cos\gamma\sin\beta & \cos\beta \end{bmatrix} \tag{24}$$

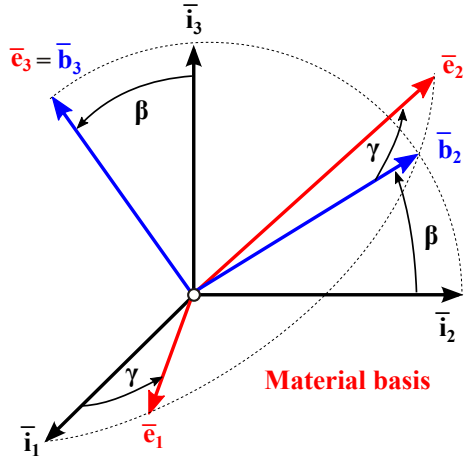

Figure 3: Orientation of the material basis using the "global axes option."

Note that positive angles β and γ correspond to positive rotations about axes $\bar{\imath}_1$ and $\bar{b}_3$, respectively, following the right-hand rule. It is important to note that in this scheme, while the layer orientation depends on the solid local basis, the determination of the material basis orientation is independent of that of the solid local basis.

4.3 Rotating material properties

Let the components of the stress tensor resolved in bases $\mathcal{I}$ and $\mathcal{E}$ be denoted $\underline{\underline{\sigma}}$ and $\underline{\underline{\sigma}}^+$, respectively. The following relationship then holds, $\underline{\underline{\sigma}}^+ = \underline{\underline{R}}^T \underline{\underline{\sigma}}\, \underline{\underline{R}}$, where $\underline{\underline{R}}$ are the components of the rotation tensor that brings basis $\mathcal{I}$ to basis $\mathcal{E}$, resolved in basis $\mathcal{I}$ [5]. For the problem at hand, the inverse relationship is desired, $\underline{\underline{\sigma}} = \underline{\underline{R}}\, \underline{\underline{\sigma}}^+ \underline{\underline{R}}^T$.

To simplify the notation, let

$$\underline{\underline{R}} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}, \tag{25}$$

where ℓ_1, ℓ_2, and ℓ_3, m_1, m_2, and m_3, and n_1, n_2, and n_3 are the direction cosines of unit vectors $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$, respectively. The stress transformation, written in term of the stress array, see eq. (2), then becomes $\underline{\sigma} = \underline{\underline{R}}_\sigma \underline{\sigma}^+$, where

$$\underline{\underline{R}}_\sigma = \begin{bmatrix} \ell_1^2 & m_1^2 & n_1^2 & 2m_1n_1 & 2\ell_1n_1 & 2\ell_1m_1 \\ \ell_2^2 & m_2^2 & n_2^2 & 2m_2n_2 & 2\ell_2n_2 & 2\ell_2m_2 \\ \ell_3^2 & m_3^2 & n_3^2 & 2m_3n_3 & 2\ell_3n_3 & 2\ell_3m_3 \\ \ell_2\ell_3 & m_2m_3 & n_2n_3 & m_2n_3+m_3n_2 & \ell_2n_3+\ell_3n_2 & \ell_2m_3+\ell_3m_2 \\ \ell_1\ell_3 & m_1m_3 & n_1n_3 & m_1n_3+m_3n_1 & \ell_1n_3+\ell_3n_1 & \ell_1m_3+\ell_3m_1 \\ \ell_1\ell_2 & m_1m_2 & n_1n_2 & m_1n_2+m_2n_1 & \ell_1n_2+\ell_2n_1 & \ell_1m_2+\ell_2m_1 \end{bmatrix}. \tag{26}$$

The strain transformation, written in term of the strain array, see eq. (1), becomes $\underline{\epsilon} = \underline{\underline{R}}_\epsilon \underline{\epsilon}^+$, where

$$\underline{\underline{R}}_\epsilon = \begin{bmatrix} \ell_1^2 & m_1^2 & n_1^2 & m_1n_1 & \ell_1n_1 & \ell_1m_1 \\ \ell_2^2 & m_2^2 & n_2^2 & m_2n_2 & \ell_2n_2 & \ell_2m_2 \\ \ell_3^2 & m_3^2 & n_3^2 & m_3n_3 & \ell_3n_3 & \ell_3m_3 \\ 2\ell_2\ell_3 & 2m_2m_3 & 2n_2n_3 & m_2n_3+m_3n_2 & \ell_2n_3+\ell_3n_2 & \ell_2m_3+\ell_3m_2 \\ 2\ell_1\ell_3 & 2m_1m_3 & 2n_1n_3 & m_1n_3+m_3n_1 & \ell_1n_3+\ell_3n_1 & \ell_1m_3+\ell_3m_1 \\ 2\ell_1\ell_2 & 2m_1m_2 & 2n_1n_2 & m_1n_2+m_2n_1 & \ell_1n_2+\ell_2n_1 & \ell_1m_2+\ell_2m_1 \end{bmatrix}. \tag{27}$$

The constitutive laws, written as $\underline{\sigma}^+ = \underline{\underline{C}}^+ \underline{\epsilon}^+$ in eq. (3a) can now be transformed: $\underline{\sigma} = \underline{\underline{R}}_\sigma \underline{\sigma}^+ = \underline{\underline{R}}_\sigma \underline{\underline{C}}^+ \underline{\underline{R}}_\epsilon^{-1} \underline{\epsilon}$, which implies that $\underline{\underline{C}} = \underline{\underline{R}}_\sigma \underline{\underline{C}}^+ \underline{\underline{R}}_\epsilon^{-1}$. Because the stiffness matrix is symmetric in all bases, $\underline{\underline{R}}_\epsilon^{-1} = \underline{\underline{R}}_\sigma^T$, and hence,

$$\underline{\underline{C}} = \underline{\underline{R}}_\sigma \underline{\underline{C}}^+ \underline{\underline{R}}_\sigma^T. \tag{28}$$

5 Material viscoelasticity properties

5.1 Generalized Maxwell model

The physical viscoelastic properties of materials are defined in this section. These properties are based on the Generalized Maxwell model [6].

5.1.1 One-dimensional generalized Maxwell model

One-dimensional rheological models [7, 8] are often used to introduce the concepts associated with viscoelasticity. Typically, these models involve serial or parallel combinations of linear or nonlinear springs and dashpots to form increasingly complex models, such as the Kelvin, Maxwell, or Zener models, among many others. For instance, fig. 4 depicts the generalized Maxwell model. The spring elements characterize the elastic behavior of the material, whereas its energy dissipation characteristics are described by the dashpot elements.

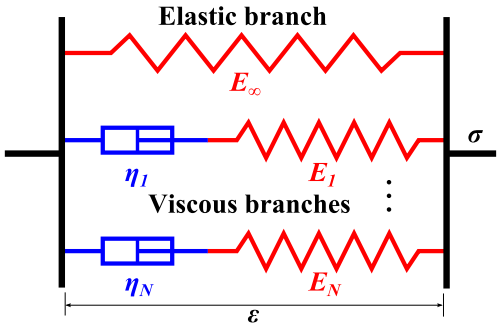

Figure 4: Schematic of the generalized Maxwell model.

The generalized Maxwell model depicted schematically in fig 4 consists of an elastic spring (the "elastic branch") in parallel with one or more Maxwell fluid elements (the "viscous branches"). It generalizes the Zener solid model, also known as the *standard linear solid* model, which possesses a single elastic branch in parallel with a single viscous branch. Assuming the device to be of unit area and length, forces and elongations can be identified with stresses and strains, respectively. The spring stiffness constants are denoted $E_\infty > 0$ and $E_b > 0$, $b = 1, \ldots, N_b$, where N_b is the number of Maxwell fluid elements. The dashpot constants are denoted $\eta_b > 0$, $b = 1, \ldots, N_b$.

Elementary mechanics yields the constitutive equations for the various components of the system. For the elastic branch, $\sigma_\infty = E_\infty \epsilon$, where σ_∞ is the stress in the elastic branch and ϵ the device strain. The stress in a typical viscous branch is $\sigma_b = \eta_b \dot{\alpha}_b = E_b(\epsilon - \alpha_b)$, where the first equation provides the constitutive equation for the dashpot, the second that for the elastic spring, and α_b can be interpreted as the strain in the dashpot. Notation $(\cdot)^{\cdot}$ indicates a derivative with respect to time. Finally, the total stress, σ, is found by summing up the stresses in the branches, $\sigma = \sigma_\infty + \sum_{b=1}^{N_b} \sigma_b$. Introducing the constitutive relationships yields

$$\sigma = E_\infty \epsilon + \sum_{b=1}^{N_b} E_b(\epsilon - \alpha_b), \tag{29}$$

where α_b are internal strain states of the model.

Because the stresses in the spring and dashpot of each viscous branch are identical, $\eta_b \dot{\alpha}_b = E_b(\epsilon - \alpha_b)$, and hence, the internal states must satisfy the following evolution equation

$$\tau_b \dot{\alpha}_b + \alpha_b = \epsilon, \quad b = 1, 2, \ldots, N_b, \tag{30}$$

where the *relaxation times*, τ_b, are defined as

$$\tau_b = \frac{\eta_b}{E_b}. \tag{31}$$

The first-order differential evolution equation (30) must satisfy the following initial condition, $\lim_{t \to -\infty} \alpha_b = 0$.

Because the evolution equation (30) can be recast as $\mathrm{d}[\exp(t/\tau_b)\alpha_b]/\mathrm{d}t = (1/\tau_b)\exp(t/\tau_b)\epsilon$, integration over time yields $\alpha_b = (1/\tau_b)\int_{-\infty}^{t} \exp[-(t-s)/\tau_b]\epsilon(s)\ \mathrm{d}s$. Assuming that $\epsilon(t) \to 0$ as $t \to -\infty$, integration by parts leads to

$$\alpha_b(t) = \epsilon(t) - \int_{-\infty}^{t} e^{-(t-s)/\tau_b} \dot{\epsilon}(s)\ \mathrm{d}s. \tag{32}$$

The stress convolution integral then follows from introducing eq. (32) into eq. (29) to find

$$\sigma(t) = \int_{-\infty}^{t} G(t-s)\dot{\epsilon}(s)\ \mathrm{d}s, \tag{33}$$

where the relaxation function, $G(t)$, is defined as

$$G(t) = E_\infty + \sum_{b=1}^{N_b} E_b e^{-t/\tau_b}. \tag{34}$$

In summary, the one-dimensional generalized Maxwell model can be expressed by convolution integral (33), where the relaxation function is defined by eq. (34). Alternatively, the same model can be represented by eq. (29), where the internal variables satisfy the differential equations of evolution (30). These two formulations of the same model will be used in the sequel.

5.1.2 Three-dimensional generalized Maxwell model

Following the suggested approach of Simo and Hughes [6], the one-dimensional constitutive law for viscoelastic material, eq. (33), can be extended to three-dimensional form by writing the following convolution integral,

$$\underline{\sigma}^+(t) = \int_{-\infty}^{t} \underline{\underline{G}}^+(t-s)\dot{\underline{\gamma}}^+(s)\, \mathrm{d}s, \tag{35}$$

where arrays $\underline{\sigma}^+$ and $\underline{\gamma}^+$ store the six components of the convected Cauchy stress and Green-Lagrange strain tensors, respectively, both resolved in the material basis and the relaxation matrix function, $\underline{\underline{G}}^+(t)$, echoes its corresponding scalar equivalent defined by eq. (34),

$$\underline{\underline{G}}^+(t) = \underline{\underline{\mathcal{C}}}^+_\infty + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}^+_b e^{-t/\tau_b}. \tag{36}$$

Relaxation times, τ_b, $b = 1, 2, \ldots, N_b$, and matrices $\underline{\underline{\mathcal{C}}}^+_\infty$ and $\underline{\underline{\mathcal{C}}}^+_b$, $b = 1, 2, \ldots, N_b$, characterize the three-dimensional viscoelastic behavior of the material. Stiffness matrix $\underline{\underline{\mathcal{C}}}^+_\infty$ characterizes the elastic behavior of the material and echoes the elastic constitutive law (3a). Note that the viscoelastic material behavior described by eq. (35) reduces to the elastic behavior described by eq. (3a) if $N_b = 0$ or equivalently, for vanishing relaxation times, $\tau_b \to 0$.

Elementary mathematical arguments show that the convolution integral formulation shown in the previous paragraph can be recast in terms of ordinary differential equations. The total stresses become

$$\underline{\sigma}^+(t) = \underline{\underline{\mathcal{C}}}^+_\infty \underline{\gamma}^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}^+_b \left(\underline{\gamma}^+(t) - \underline{\alpha}^+_b(t)\right), \tag{37}$$

where the internal variables of the model, $\underline{\alpha}^+_b$, satisfy the following matrix evolution equations

$$\tau_b \dot{\underline{\alpha}}^+_b + \underline{\alpha}^+_b = \underline{\gamma}^+, \quad b = 1, 2, \ldots, N_b, \tag{38}$$

with the following initial condition $\lim_{t\to-\infty} \underline{\alpha}^+_b(t) = \underline{0}$. Note the direct parallel between equations (37) and (38), written for three-dimensional problems and their counterparts, equations (29) and (30), respectively, written for one-dimensional problems.

5.1.3 Time domain description

Equation (35) describes the behavior of viscoelastic materials in the time domain when subjected to a strain history, $\underline{\gamma}^+(t)$. The response of the material is characterized by the relaxation function, $\underline{\underline{G}}^+(t)$, which can be interpreted as the time-dependent stiffness matrix of the material. In the generalized Maxwell model, the relaxation function is expanded in Prony series, see eq. (36).

The physical interpretation of the relaxation function is obtained easily. Consider a step input in strain history, $\underline{\gamma}^+(t) = 0$ for $t < 0$, and $\underline{\gamma}^+(t) = \underline{\gamma}^+$ for $t \geq 0$. Introducing this strain history in eq. (35) yields

$$\underline{\sigma}^+(t) = \int_{-\infty}^{t} \underline{\underline{G}}^+(t-s)\dot{\underline{\gamma}}^+(s)\, \mathrm{d}s = \underline{\underline{G}}^+(t)\underline{\gamma}^+. \tag{39}$$

Clearly, the relaxation function describes the stress history after the application of the step input in strain. The instantaneous stiffness matrix at time $t = 0$ is $\underline{\underline{\mathcal{C}}}_0^+ = \underline{\underline{\mathcal{C}}}_\infty^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+$, whereas the long-term steady stiffness matrix is $\underline{\underline{\mathcal{C}}}_\infty^+$ when $t \to \infty$.

In practice, material viscoelastic properties can be obtained from a stress relaxation experiment: a strain step input is applied to the material and the measurement of the resulting stress provides the stress relaxation function. Curve fitting techniques are then used to extract Prony series from the experimental data.

5.1.4 Relaxation function in frequency domain

In contrast to the time domain characterization described in the previous section, viscoelastic material can also be described in the frequency domain. Consider the following harmonic strain input,

$$\underline{\gamma}^+(t) = \underline{\gamma}_0^+ e^{i\omega t}, \tag{40}$$

where ω is the excitation frequency and $i = \sqrt{-1}$. Introducing eqs. (40) and (36) into eq. (35) yields

$$\begin{aligned} \underline{\sigma}^+(t) &= \int_{-\infty}^{t} (\underline{\underline{\mathcal{C}}}_\infty^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ e^{-(t-s)/\tau_b})(i\omega \underline{\gamma}_0^+ e^{i\omega s})\, \mathrm{d}s \\ &= \left[\underline{\underline{\mathcal{C}}}_\infty^+ + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ \frac{i(\omega\tau_b)}{1 + i(\omega\tau_b)}\right] \underline{\gamma}^+(t). \end{aligned} \tag{41}$$

It is customary to characterize the harmonic response of the material in terms of the storage and loss stiffness matrices, denoted $\underline{\underline{G}}_s(\omega)$ and $\underline{\underline{G}}_\ell(\omega)$, respectively,

$$\underline{\sigma}^+(t) = \left[\underline{\underline{G}}_s(\omega) + i\underline{\underline{G}}_\ell(\omega)\right] \underline{\gamma}^+(t), \tag{42}$$

which characterize the in-phase and out-of-phase stress response. Identifying eqs. (41) and (42) then leads to

$$\begin{aligned} \underline{\underline{G}}_s(\omega) &= \underline{\underline{\mathcal{C}}}_\infty + \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ \frac{(\omega\tau_b)^2}{1 + (\omega\tau_b)^2} \\ \underline{\underline{G}}_\ell(\omega) &= \sum_{b=1}^{N_b} \underline{\underline{\mathcal{C}}}_b^+ \frac{(\omega\tau_b)}{1 + (\omega\tau_b)^2}. \end{aligned} \tag{43}$$

Experimentally, the storage and loss matrices are obtained by dynamic mechanical analysis (DMA). Curve fitting techniques are then used to approximate the experimental data by eq. (43).

5.2 Simple viscoelastic elements

This section contrasts the characteristics of simple viscoelastic elements: the Kelvin-Voigt, the Maxwell fluid element, and the Zener element, in sections 5.2.1, 5.2.2, and 5.2.3, respectively.

5.2.1 The Kelvin-Voigt element

As depicted in fig. 5, the Kelvin-Voigt element consists of an elastic spring of constant E, and a dashpot of constant η arranged in parallel. The constitutive laws for the two components are $\sigma_e = E\varepsilon$ and $\sigma_v = \eta\dot{\varepsilon}$, where σ_e and σ_v are the stresses in the spring and dashpot, respectively. The total stress in the Kelvin-Voigt element is now

$$\sigma = \sigma_e + \sigma_v = E\varepsilon + \eta\dot{\varepsilon}. \tag{44}$$

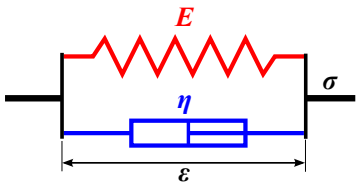

Figure 5: Schematic of the Kelvin-Voigt element

It is customary to describe the characteristics of viscoelastic elements in the frequency domain; let $\varepsilon(t) = \varepsilon_0 \exp(i\omega t)$, where ε_0 is the amplitude of the excitation and ω its frequency. The stress in the element is now

$$\sigma = E(1 + i\omega\tau)\varepsilon, \tag{45}$$

where $\tau = \eta/E$ is the relaxation time. Stiffness modulus $G_e + iG_v = E(1 + i\omega\tau)$ is called the "complex modulus" of the Kelvin-Voigt element, where G_e and G_ℓ are the "elastic modulus" and "loss modulus," respectively. Figures 6 and 7 show the non-dimensional elastic and loss moduli, defined as $\bar{G}_e = G_e/E = 1$ and $\bar{G}_\ell = G_\ell/E = \omega\tau$, versus the non-dimensional excitation frequency, $\omega\tau$, on a logarithmic scale.

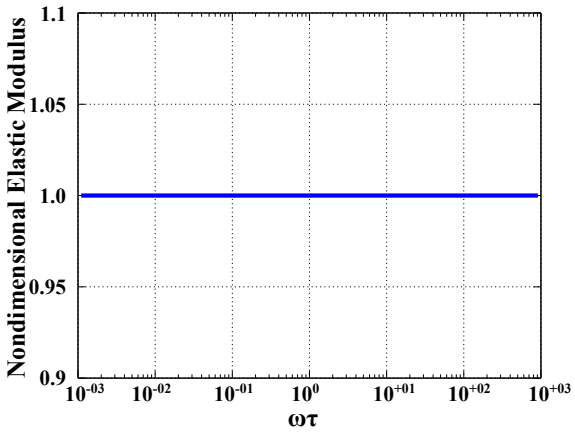

Figure 6: Non-dimensional elastic modulus for the Kelvin-Voigt element versus non-dimensional frequency, $\omega\tau$, on a logarithmic scale.

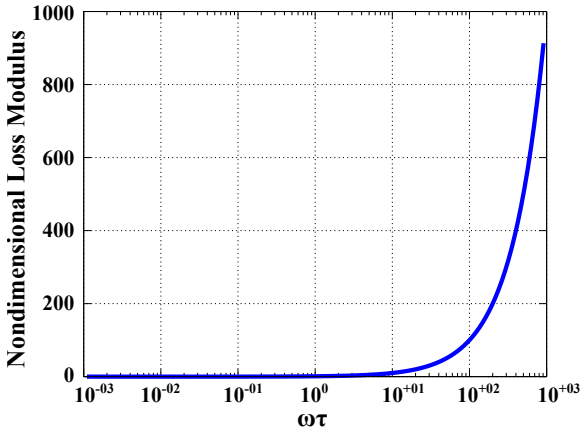

Figure 7: Non-dimensional elastic modulus for the Kelvin-Voigt element versus non-dimensional frequency, $\omega\tau$, on a logarithmic scale.

The viscous stress in the Kelvin-Voigt element is $\sigma_v = \eta\dot{\varepsilon}$ and the work done by this force over one period of harmonic oscillation is then

$$W_v = \int_0^T \eta\dot{\varepsilon}\dot{\varepsilon}\, \mathrm{d}t = \pi(\omega\tau)E\varepsilon_0^2, \tag{46}$$

where $T = 2\pi/\omega$ is the period of the excitation. The non-dimensional work done by the viscous force is now

$$\bar{W}_v = \frac{W_v}{\pi E\varepsilon_0^2} = \omega\tau. \tag{47}$$

Note that the non-dimensional work done by the viscous force over one period of excitation equals the non-dimensional loss modulus. Consequently, fig. 7 can also be interpreted as the non-dimensional work done by the viscous force over one period of excitation versus non-dimensional excitation frequency. Note that this work rises rapidly as the excitation frequency increases.

5.2.2 The Maxwell fluid element

Figure 8 depicts the Maxwell fluid element, which consists of an elastic spring of constant E and a dashpot of constant η arranged in series. Let $\alpha(t)$ denote the strain in the dashpot; the constitutive laws for the two components are $\sigma = E(\varepsilon - \alpha)$ and $\sigma = \eta\dot{\alpha}$, where σ denotes the stress in the element. The governing equations for the Maxwell fluid element are now

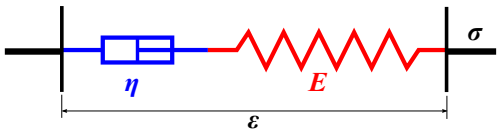

Figure 8: Schematic of the Maxwell element

$$\begin{aligned} \tau\dot{\alpha} &= \varepsilon - \alpha, \\ \sigma &= E(\varepsilon - \alpha), \end{aligned} \tag{48}$$

where $\tau = \eta/E$ is the relaxation time. It is assumed that $\sigma = 0$ for $t \to -\infty$.

Let $\varepsilon(t) = \varepsilon_0 \exp(i\omega t)$, where ε_0 is the amplitude of the excitation and ω its frequency. Equations (48) are solved in the frequency domain easily, to find

$$\frac{\sigma}{E} = \left[\frac{(\omega\tau)^2}{1+(\omega\tau)^2} + i\frac{(\omega\tau)}{1+(\omega\tau)^2}\right]\varepsilon. \tag{49}$$

Figures 9 and 10 show the non-dimensional elastic and loss moduli, defined as $\bar{G}_e = G_e/E = (\omega\tau)^2/[1+(\omega\tau)^2]$ and $\bar{G}_\ell = G_\ell/E = (\omega\tau)/[1+(\omega\tau)^2]$, versus the non-dimensional excitation frequency, $\omega\tau$, on a logarithmic scale.

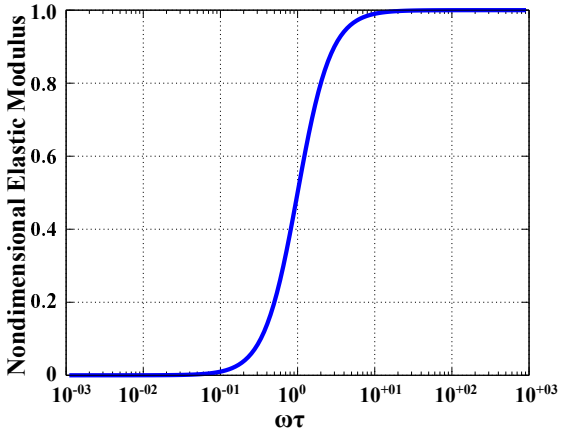

Figure 9: Non-dimensional elastic modulus for the Maxwell fluid element versus non-dimensional frequency, $\omega\tau$, on a logarithmic scale.

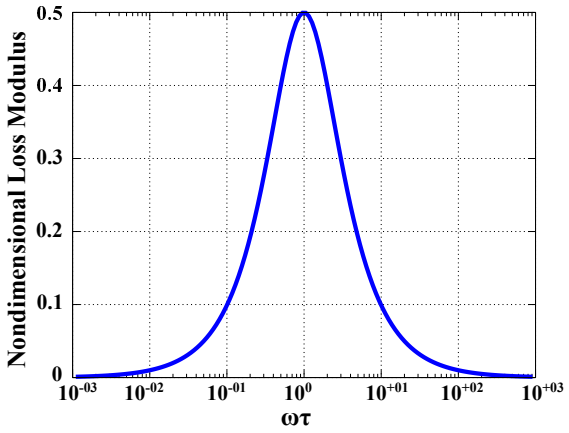

Figure 10: Non-dimensional elastic modulus for the Maxwell fluid element versus non-dimensional frequency, $\omega\tau$, on a logarithmic scale.

Figure 9 shows that at low excitation frequencies, *i.e.*, when $\omega\tau \to 0$, the elastic modulus of the Maxwell fluid element vanishes, as expected for a fluid. In contrast, the elastic modulus of the Kelvin-Voigt element is independent of frequency, see fig. 6.

The loss moduli of the Kelvin-Voigt and Maxwell fluid elements also differ sharply, as depicted in figs. 7 and 10. Note that the loss modulus of the Kelvin-Voigt element increases without bounds at high excitation frequencies, in contrast with that of the Maxwell fluid element, which shows a maximum for $\omega\tau = 1$, then decreases at high frequencies again. This latter behavior is that observed in actual materials. Figure 11 compares the loss moduli of the

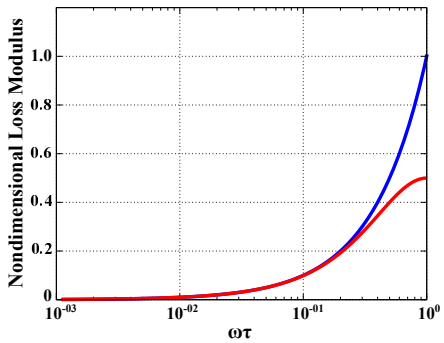

Figure 11: Comparison of the loss moduli of

Kelvin-Voigt and Maxwell fluid elements at low frequencies. Note that at low excitation frequencies, $\omega\tau < 0.2$, the two loss moduli are in close agreement with each other. Consequently, the Kelvin-Voigt model should be used at low frequencies only.

5.2.3 The Zener solid element

Figure 12 depicts the Zener solid element, also known as the "linear solid element," which consists of an elastic spring of constant E_∞ and a Maxwell fluid element arranged in series. The physical characteristics of the Maxwell fluid element are E_1 and η_1, as indicated in the figure. Let $\alpha(t)$ denote the strain in the dashpot. The governing equations for the Zener solid element are now

$$\begin{aligned} \tau\dot{\alpha} &= \varepsilon - \alpha, \\ \sigma &= E_\infty\varepsilon + E_1(\varepsilon - \alpha), \end{aligned} \tag{50}$$

where $\tau = \eta/E$ is the relaxation time. It is assumed that $\sigma = 0$ for $t \to -\infty$. Note that for $E_\infty = 0$, the Zener solid element degenerates into the Maxwell fluid element.

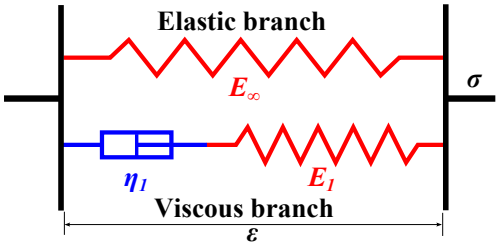

Figure 12: Schematic of the Maxwell element.

Let $\varepsilon(t) = \varepsilon_0 \exp(i\omega t)$, where ε_0 is the amplitude of the excitation and ω its frequency. Equations (50) are solved in the frequency domain easily, to find

$$\frac{\sigma}{E_\infty + E_1} = \left[\frac{(1-p) + (\omega\tau)^2}{1 + (\omega\tau)^2} + i\frac{p(\omega\tau)}{1 + (\omega\tau)^2}\right]\varepsilon, \tag{51}$$

where $p = E_1/(E_\infty + E_1)$.

Figures 13 and 14 show the non-dimensional elastic and loss moduli, defined as $\bar{G}_e = G_e/(E_\infty + E_1) = [(1-p) + (\omega\tau)^2]/[1 + (\omega\tau)^2]$ and $\bar{G}_\ell = G_\ell/(E_\infty + E_1) = p(\omega\tau)/[1 + (\omega\tau)^2]$, versus the non-dimensional excitation frequency, $\omega\tau$, on a logarithmic scale. Five values of ration p are illustrated in the figures, $p = 0.01$, 0.25, 0.50, 0.75, and 1.0. Note again that for $p = 1$, the Zener solid element degenerates into the Maxwell fluid element.

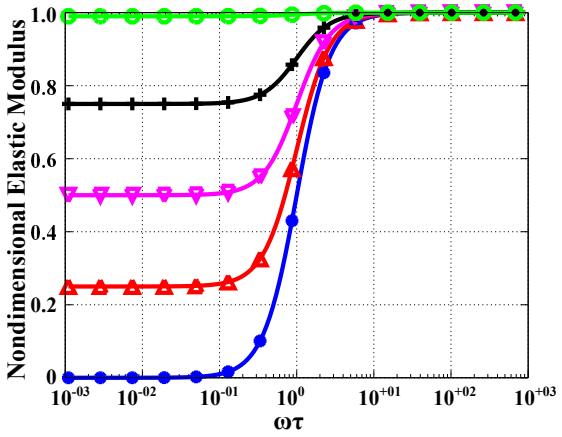

Figure 13: Non-dimensional elastic modulus for the Zener solid element versus non-dimensional frequency, $\omega\tau$, on a logarithmic scale. $p = 0.01$ (green), 0.25 (black), 0.50 (magenta), 0.75 (red), and 1.0 (blue).

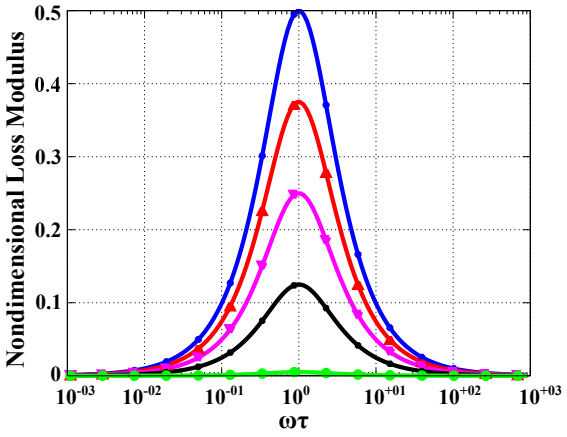

Figure 14: Non-dimensional elastic modulus for the Zener solid element versus non-dimensional frequency, $\omega\tau$, on a logarithmic scale. $p = 0.01$ (green), 0.25 (black), 0.50 (magenta), 0.75 (red), and 1.0 (blue).

As was the case for the Maxwell fluid element, the loss modulus of the Zener solid element vanishes at both low and high frequencies, *i.e.*, when $(\omega\tau) \to 0$ and $(\omega\tau) \to \infty$. For the Zener solid element, low loss moduli are obtained for small values of its characteristic parameters, *i.e.*, for $p \to 0$ and $\tau \to 0$. The same conclusions are reached for the three-dimensional version of the Zener solid model.

5.3 Branch definitions

As depicted in fig. 4, the general Maxwell model contains an elastic branch and N_b viscous branches. The elastic branch properties are defined based on the section of **materials stiffness properties**, while the viscous branches properties are defined according to the **branch definitions** of the material viscoelasticity properties.

Each viscous branch has two characteristic properties, a relaxation time τ_b and a characteristic stiffness matrix $\underline{\underline{\mathcal{C}}}_b$. Matrices $\underline{\underline{\mathcal{C}}}_b$ are defined in different form according to the material symmetric types. Three types of linear viscoelastic materials are considered here, *isotropic*, *orthotropic* and *transversely isotropic* materials.

5.3.1 Orthotropic viscoelastic materials

Not implemented at this time.

5.3.2 Transversely isotropic viscoelastic materials

Not implemented at this time.

5.3.3 Isotropic viscoelastic materials

An *isotropic material* is a material that presents identical properties in all directions. For these materials, two independent parameters are needed to decide the stiffness matrix [7]. Because the shear behavior and bulk behavior are usually considered in viscoelasticity, the stiffness matrix defined here takes the following form,

$$\underline{\underline{\mathcal{C}}}_b^+ = \begin{bmatrix} (3\xi_b + 4\eta_b)/3 & (3\xi_b - 2\eta_b)/3 & (3\xi_b - 2\eta_b)/3 & 0 & 0 & 0 \\ (3\xi_b - 2\eta_b)/3 & (3\xi_b + 4\eta_b)/3 & (3\xi_b - 2\eta_b)/3 & 0 & 0 & 0 \\ (3\xi_b - 2\eta_b)/3 & (3\xi_b - 2\eta_b)/3 & (3\xi_b + 4\eta_b)/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_b & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_b & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_b \end{bmatrix}. \tag{52}$$

Hence, for isotropic materials, the following *three* properties are required for each viscous branch: *(1)* **Relaxation time:** τ_b, *(2)* **Bulk modulus:** ξ_b, *(3)* **Shear modulus:** η_b.

These three required parameters can be obtained by material experiments. The **time domain** and the **frequency domain** experiments are introduced here.

In **time domain** if a relaxation experiment in shear behavior is performed, according to equations (36) and (39), the time-dependent shear modulus becomes,

$$\eta(t) = \eta_\infty + \sum_{b=1}^{N_b} \eta_b e^{-t/\tau_b}, \tag{53}$$

where η_∞ is the equivalent shear modulus defined in eq. (9). Time-dependent shear modulus can be obtained by performing the relaxation experiment. The time-dependent shear modulus can then

be approximated by a number of Prony series according to eq. (53), which can be fitted with the experiment data via different data fitting methods, such as least squares. Each Prony term is defined in one viscous branch. In each branch the relaxation time for shear and bulk behavior is assumed to be the same in Dymore. The method to decide the bulk modulus in time domain is similar to that of shearing.

In **frequency domain** if a dynamic mechanical analysis in shear behavior is performed, according to eq. (41), the shear response becomes,

$$\sigma(t) = [\eta_s(\omega) + i\eta_l(\omega)]\,\gamma(t), \tag{54}$$

where $\gamma(t)$ is the harmonic shear strain, $\sigma(t)$ is the shear stress response, $\eta_s(\omega)$ is the storage shear modulus and $\eta_l(\omega)$ is the loss shear modulus.

The absolute magnitude of the shear response is

$$|\sigma| = \sqrt{\eta_s^2(\omega) + \eta_l^2(\omega)}|\gamma|, \tag{55}$$

and the phase lag of the shear response is

$$\phi = \arctan\frac{\eta_l(\omega)}{\eta_s(\omega)}. \tag{56}$$

Measurements of $|\tau|$ and ϕ obtained in the experiment can then be used to define η_s and η_l [9]. Then the storage shear modulus and loss shear modulus can be expressed parallel to eq. (43),

$$\begin{aligned}\eta_s(\omega) &= \eta_\infty + \sum_{b=1}^{N_b} \eta_b^+ \frac{\omega^2\tau_b^2}{1+\omega^2\tau_b^2}\\ \eta_l(\omega) &= \sum_{b=1}^{N_b} \eta_b^+ \frac{\omega\tau_b}{1+\omega^2\tau_b^2}.\end{aligned} \tag{57}$$

In each branch the relaxation time for shear and bulk response is assumed to be the same in Dymore. The method for bulk behavior is also analogous to that of the shear behavior.

In general the three required parameters can be obtained from relaxation experiment according to eq. (53), or from harmonic dynamic mechanical analysis according to eq. (57).

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