Dymore User's Manual

Basic Geometric entities

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Contents

1 Points

The simplest way to represent the location of a point in three-dimensional space is to make use of a reference frame, $\mathcal{F}_I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$, consisting of an orthonormal basis $\mathcal{I}$ with its origin and point $\mathbf{O}$. The position vector of point $\mathbf{P}$ is represented by its *Cartesian coordinates*, x_1, x_2, and x_3, resolved along unit vectors, $\bar{\imath}_1$, $\bar{\imath}_2$, and $\bar{\imath}_3$, respectively,

$$\underline{r} = x_1 \bar{\imath}_1 + x_2 \bar{\imath}_2 + x_3 \bar{\imath}_3, \tag{1}$$

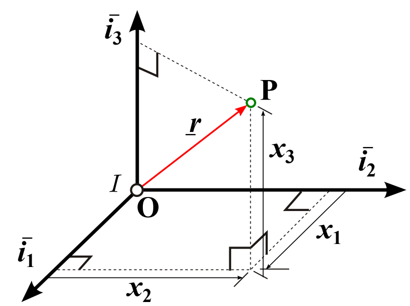

Figure 1: Configuration of a point.

where t denotes time. Figure 1 depicts the situation: Cartesian coordinate $x_1 = \bar{\imath}_1^T \underline{r}$ is the projection of the position vector of point $\mathbf{P}$ along unit vector $\bar{\imath}_1$. Similarly, Cartesian coordinates x_1 and x_2 are the projections of the same position vector along unit vectors $\bar{\imath}_2$ and $\bar{\imath}_3$, respectively.

2 Triads

Figure 2 shows the inertial frame, $\mathcal{F}_I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$, and an orthonormal triad, $\mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$. For clarity of the figure, the triad is offset with respect to the origin of the inertial frame. A *rotation* is defined as the operation that brings basis $\mathcal{I}$ to basis $\mathcal{E}$. Unit vector $\bar{e}_1$ can be expressed as a linear combination of the vectors of basis $\mathcal{I}$

$$\bar{e}_1 = D_{11} \bar{\imath}_1 + D_{21} \bar{\imath}_2 + D_{31} \bar{\imath}_3. \tag{2}$$

The coefficients of this linear combination are readily expressed as $D_{k1} = \bar{\imath}_k^T \bar{e}_1$. Proceeding similarly with the three unit vectors defining basis $\mathcal{E}$ yields the terms of the *direction cosine matrix*, $\underline{\underline{D}}$, as

$$D_{k\ell} = \bar{\imath}_k^T \bar{e}_\ell. \tag{3}$$

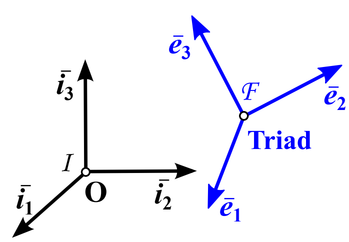

Figure 2: Configuration of a triad.

Observing that vectors $\bar{\imath}_k$ and $\bar{e}_\ell$ are unit vectors yields an alternative expression for the direction cosine matrix

$$D_{kl} = \cos\left(\bar{\imath}_k, \bar{e}_\ell\right). \tag{4}$$

This expression gives its name to the direction cosine matrix: its entries are the cosine of the angle between $\bar{\imath}_k$ and $\bar{e}_\ell$, the unit vectors defining bases $\mathcal{I}$ and $\mathcal{E}$, respectively. Each component of the direction cosine matrix is a scalar quantity. The direction cosine matrix, however, is not a second-order tensor.

A triad can be defined in many different manner described in the following sections.

2.1 Two vectors

The orientation of the triad can be defined by the components of two vectors, $\underline{v}_2$ and $\underline{v}_3$, resolved in inertial basis $\mathcal{I}$, as depicted in fig. 3. Vector $\underline{v}_2$ defines the orientation of unit vector $\bar{e}_2$ and plane $(\underline{v}_2, \underline{v}_3)$ is identical to plane $(\bar{e}_2, \bar{e}_3)$. A valid definition requires the satisfaction of the following two conditions: *(1)* $\|\underline{v}_2\| \neq 0$ and *(2)* $\tilde{v}_2 \underline{v}_3 \neq \underline{0}$.

The following algorithm details the procedure for the evaluation of unit vectors $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$. *(1)* First, unit vector $\bar{e}_2$ is computed by normalizing vector $\underline{v}_2$ to find

$$\bar{e}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|}. \tag{5}$$

(2) Next, unit vector $\bar{e}_3$ is computed by orthogonalizing vector $\underline{v}_3$ against $\bar{e}_2$, then normalizing the result, leading to

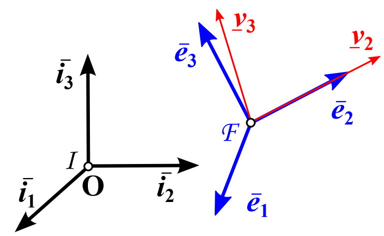

Figure 3: Construction of a triad.

$$\underline{v}_3' = \underline{v}_3 - (\bar{e}_2^T \underline{v}_3)\bar{e}_2, \quad \bar{e}_3 = \frac{\underline{v}_3'}{\|\underline{v}_3'\|}. \tag{6}$$

(3) Finally, unit vector $\bar{e}_1$ is the vector product of unit vectors $\bar{e}_2$ and $\bar{e}_3$,

$$\bar{e}_1 = \tilde{e}_2 \bar{e}_3. \tag{7}$$

2.2 Three points

The orientation of the triad can be defined by the components of three points, $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$, as shown in fig. 4. Let $\underline{v}_1 = \mathbf{P_2} - \mathbf{P_1}$ and $\underline{v}_2 = \mathbf{P_3} - \mathbf{P_1}$. Vector $\underline{v}_1$ gives the orientation of unit vector $\bar{e}_1$ and plane $(\underline{v}_1, \underline{v}_2)$ is identical to plane $(\bar{e}_1, \bar{e}_2)$. A valid definition requires the satisfaction of the following two conditions: *(1)* $\|\underline{v}_1\| \neq 0$ and *(2)* $\tilde{v}_1 \underline{v}_2 \neq \underline{0}$.

The following algorithm details the procedure for the evaluation of unit vectors $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$. *(1)* Unit vector $\bar{e}_1$ is computed by normalization of vector $\underline{v}_1$,

$$\underline{e}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}. \tag{8}$$

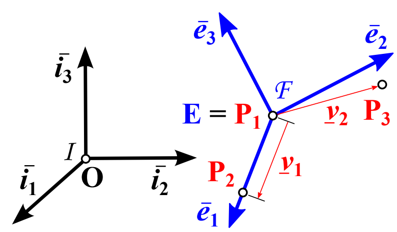

Figure 4: Construction of a triad.

(2) Unit vector $\bar{e}_2$ is computed by orthogonalizing vector $\underline{v}_2$ against $\bar{e}_1$, then normalizing the result, to find

$$\underline{v}_2' = \underline{v}_2 - (\bar{e}_1^T \underline{v}_2)\bar{e}_1; \quad \bar{e}_2 = \frac{\underline{v}_2'}{\|\underline{v}_2'\|}. \tag{9}$$

(3) Finally, unit vector $\bar{e}_3$ is the vector product of unit vectors $\bar{e}_1$ and $\bar{e}_2$,

$$\bar{e}_3 = \tilde{e}_1 \bar{e}_2. \tag{10}$$

2.3 A single vector

The orientation of the triad can be defined by the components of a single vector, say $\underline{v}_3$, resolved in inertial basis $\mathcal{I}$, as depicted in fig. 5. Vector $\underline{v}_3$ defines the orientation of unit vector $\bar{e}_3$. Clearly, this definition of the triad is ambiguous because an arbitrary planar rotation about vector $\underline{v}_3$ leaves the direction of this vector unchanged. A valid definition requires the satisfaction of the following condition: $\|\underline{v}_3\| \neq 0$.

When expressed in terms of Euler parameters [1], the rotation tensor takes the following form

$$\underline{\underline{R}} = \begin{bmatrix} 1 - 2e_2^2 - 2e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & 1 - 2e_1^2 - 2e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & 1 - 2e_1^2 - 2e_2^2 \end{bmatrix}, \tag{11}$$

where the parameters satisfy the normality condition, $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$.

2.3.1 The first vector is given

Assume the components of vector $\underline{v}_1$ are given. Unit vector $\bar{e}_1 = \bar{n} = \underline{v}_1/\|\underline{v}_1\|$ is defined first and should equal the first column of the rotation tensor defined by eq. (11). Hence,

$$\left\{\begin{matrix} 1-2e_2^2-2e_3^2 \\ 2(e_1e_2+e_0e_3) \\ 2(e_1e_3-e_0e_2) \end{matrix}\right\} = \left\{\begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix}\right\}, \tag{12}$$

where $\bar{n}^T = \{n_1, n_2, n_3\}$. To remove the indeterminacy of the problem and avoid singularities, the following algorithm is adopted.

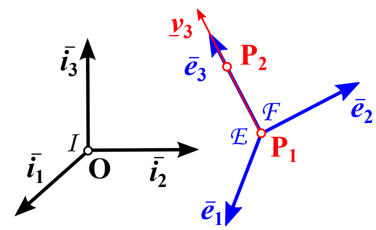

Figure 5: Construction of a triad.

- If $n_1 > 0$, it is assumed that $e_1 = e_0$. Solving eq. (12) then yields

$$\left\{\begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{matrix}\right\} = \frac{1}{2\sqrt{1+n_1}} \left\{\begin{matrix} 1+n_1 \\ 1+n_1 \\ n_2-n_3 \\ n_2+n_3 \end{matrix}\right\} \Rightarrow \underline{\underline{R}} = \begin{bmatrix} n_1 & -n_3 & n_2 \\ n_2 & -n_2n_3/(1+n_1) & n_2^2/(1+n_1)-1 \\ n_3 & 1-n_3^2/(1+n_1) & n_2n_3/(1+n_1) \end{bmatrix}. \tag{13}$$

- If $n_1 \le 0$, it is assumed that $e_2 = e_3$. Solving eq. (12) then yields

$$\left\{\begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{matrix}\right\} = \frac{1}{2\sqrt{1-n_1}} \left\{\begin{matrix} n_2-n_3 \\ n_2+n_3 \\ 1-n_1 \\ 1-n_1 \end{matrix}\right\} \Rightarrow \underline{\underline{R}} = \begin{bmatrix} n_1 & n_3 & n_2 \\ n_2 & -n_2n_3/(1-n_1) & 1-n_2^2/(1-n_1) \\ n_3 & 1-n_3^2/(1-n_1) & -n_2n_3/(1-n_1) \end{bmatrix}. \tag{14}$$

2.3.2 The second vector is given

Assume the components of vector $\underline{v}_2$ are given. Unit vector $\bar{e}_2 = \bar{n} = \underline{v}_2/\|\underline{v}_2\|$ is defined first and should equal the first column of the rotation tensor defined by eq. (11). Hence,

$$\left\{\begin{matrix} 2(e_1e_2-e_0e_3) \\ 1-2e_1^2-2e_3^2 \\ 2(e_2e_3+e_0e_1) \end{matrix}\right\} = \left\{\begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix}\right\}, \tag{15}$$

where $\bar{n}^T = \{n_1, n_2, n_3\}$. To remove the indeterminacy of the problem and avoid singularities, the following algorithm is adopted.

- If $n_2 > 0$, it is assumed that $e_2 = e_0$. Solving eq. (15) then yields

$$\left\{\begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{matrix}\right\} = \frac{1}{2\sqrt{1+n_2}} \left\{\begin{matrix} 1+n_2 \\ n_3+n_1 \\ 1+n_2 \\ n_3-n_1 \end{matrix}\right\} \Rightarrow \underline{\underline{R}} = \begin{bmatrix} n_1n_3/(1+n_2) & n_1 & 1-n_1^2/(1+n_2) \\ n_3 & n_2 & -n_1 \\ n_3^2/(1+n_2)-1 & n_3 & -n_1n_3/(1+n_2) \end{bmatrix}. \tag{16}$$

- If $n_2 \le 0$, it is assumed that $e_1 = e_3$. Solving eq. (15) then yields

$$\left\{\begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{matrix}\right\} = \frac{1}{2\sqrt{1-n_2}} \left\{\begin{matrix} n_3-n_1 \\ 1-n_2 \\ n_3+n_1 \\ 1-n_2 \end{matrix}\right\} \Rightarrow \underline{\underline{R}} = \begin{bmatrix} -n_1n_3/(1-n_2) & n_1 & 1-n_1^2/(1-n_2) \\ n_3 & n_2 & n_1 \\ 1-n_3^2/(1-n_2) & n_3 & -n_1n_3/(1-n_2) \end{bmatrix}. \tag{17}$$

2.3.3 The third vector is given

Assume the components of vector $\underline{v}_3$ are given. Unit vector $\bar{e}_3 = \bar{n} = \underline{v}_3/\|\underline{v}_3\|$ is defined first and should equal the last column of the rotation tensor defined by eq. (11). Hence,

$$\left\{\begin{matrix} 2(e_1e_3 + e_0e_2) \\ 2(e_2e_3 - e_0e_1) \\ 1 - 2e_1^2 - 2e_2^2 \end{matrix}\right\} = \left\{\begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix}\right\}, \tag{18}$$

where $\bar{n}^T = \{n_1, n_2, n_3\}$. To remove the indeterminacy of the problem and avoid singularities, the following algorithm is adopted.

- If $n_3 > 0$, it is assumed that $e_3 = e_0$. Solving eq. (18) then yields

$$\left\{\begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{matrix}\right\} = \frac{1}{2\sqrt{1+n_3}} \left\{\begin{matrix} 1+n_3 \\ n_1 - n_2 \\ n_1 + n_2 \\ 1+n_3 \end{matrix}\right\} \Rightarrow \underline{\underline{R}} = \begin{bmatrix} -n_1n_2/(1+n_3) & n_1^2/(1+n_3) - 1 & n_1 \\ 1 - n_2^2/(1+n_3) & n_1n_2/(1+n_3) & n_2 \\ -n_2 & n_1 & n_3 \end{bmatrix}. \tag{19}$$

- If $n_3 \leq 0$, it is assumed that $e_1 = e_2$. Solving eq. (18) then yields

$$\left\{\begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{matrix}\right\} = \frac{1}{2\sqrt{1-n_3}} \left\{\begin{matrix} n_1 - n_2 \\ 1-n_3 \\ 1-n_3 \\ n_1 + n_2 \end{matrix}\right\} \Rightarrow \underline{\underline{R}} = \begin{bmatrix} -n_1n_2/(1-n_3) & 1 - n_1^2/(1-n_3) & n_1 \\ 1 - n_2^2/(1-n_3) & -n_1n_2/(1-n_3) & n_2 \\ n_2 & n_1 & n_3 \end{bmatrix}. \tag{20}$$

2.4 Two points

The orientation of the triad can be defined two points, denoted points $\mathbf{P_1}$ and $\mathbf{P_2}$ in fig. 5. These two points define an orientation, $\bar{n} = (\mathbf{P_2} - \mathbf{P_1})/\|\mathbf{P_2} - \mathbf{P_1}\|$. Once this unit vector is defined, the algorithms presented in section 2.3 yield the desired triad.

2.5 Euler angles with 3-1-3 sequence

The orientation of a triad can be defined by three Euler angles [1], measured in degrees, using the *3-1-3 sequence*. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, called *precession*, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2, called *nutation*, about axis $\bar{a}_1$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3, called *spin*, about axis $\bar{b}_3$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$\underline{\underline{D}}_{3\text{-}1\text{-}3} = \begin{bmatrix} C_1C_3 - S_1C_2S_3 & -C_1S_3 - S_1C_2C_3 & S_1S_2 \\ S_1C_3 + C_1C_2S_3 & -S_1S_3 + C_1C_2C_3 & -C_1S_2 \\ S_2S_3 & S_2C_3 & C_2 \end{bmatrix}. \tag{21}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(D_{31}, D_{32}); \quad \text{if} \quad D_{32} \neq 0; \tag{22}$$

$$\phi_2 = \text{atan2}(D_{31}S_3 + D_{32}C_3, D_{33}); \tag{23}$$

$$\phi_1 = \text{atan2}(D_{21}C_3 - D_{22}S_3, D_{11}C_3 - D_{12}S_3). \tag{24}$$

It is clear that when $\theta = 0$ or π, a singularity occurs.

2.6 Euler angles with 3-2-3 sequence

The orientation of a triad can be defined by three Euler angles, measured in degrees, using the *3-2-3 sequence*. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, called *precession*, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2, called *nutation*, about axis $\bar{a}_2$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3, called *spin*, about axis $\bar{b}_3$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$\underline{\underline{D}}_{3\text{-}2\text{-}3} = \begin{bmatrix} C_1C_2C_3 - S_1S_3 & -C_1C_2S_3 - S_1C_3 & C_1S_2 \\ S_1C_2C_3 + C_1S_3 & -S_1C_2S_3 + C_1C_3 & S_1S_2 \\ -S_2C_3 & S_2S_3 & C_2 \end{bmatrix}. \tag{25}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(D_{32}, -D_{31}); \quad \text{if} \quad D_{31} \neq 0; \tag{26}$$

$$\phi_2 = \text{atan2}(D_{32}S_3 - D_{31}C_3, D_{33}); \tag{27}$$

$$\phi_1 = \text{atan2}(D_{21}C_3 - D_{22}S_3, D_{11}C_3 - D_{12}S_3). \tag{28}$$

It is clear that when $\theta = 0$ or π, a singularity occurs.

2.7 Euler angles with 3-2-1 sequence

The orientation of a triad can be defined by three Euler angles, measured in degrees, using the *3-2-1 sequence* which is popular for airplane flight mechanics. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, called *heading*, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2, called *attitude*, about axis $\bar{a}_2$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3, called *bank*, about axis $\bar{b}_1$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$\underline{\underline{D}}_{3\text{-}2\text{-}1} = \begin{bmatrix} C_1C_2 & -S_1C_3 + C_1S_2S_3 & S_1S_3 + C_1S_2C_3 \\ S_1C_2 & C_1C_3 + S_1S_2S_3 & -C_1S_3 + S_1S_2C_3 \\ -S_2 & C_2S_3 & C_2C_3 \end{bmatrix} \tag{29}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(D_{32}, D_{33}); \quad \text{if} \quad D_{33} \neq 0; \tag{30}$$

$$\phi_2 = \text{atan2}(-D_{31}, D_{32}S_3 + D_{33}C_3); \tag{31}$$

$$\phi_1 = \text{atan2}(D_{21}, D_{11}). \tag{32}$$

It is clear that when $\theta = \pm\pi/2$, a singularity occurs.

2.8 Euler angles with 3-1-2 sequence

The orientation of a triad can be defined by three Euler angles, measured in degrees, using the *3-1-2 sequence*. Euler angles, both at input or output are **measured in degrees**. This defines three consecutive planar rotations:

- A planar rotation of magnitude ϕ_1, about axis $\bar{\imath}_3$ brings $\mathcal{I}$ to $\mathcal{A}$.
- A planar rotation of magnitude ϕ_2 about axis $\bar{a}_1$ brings $\mathcal{A}$ to $\mathcal{B}$.
- A planar rotation of magnitude ϕ_3 about axis $\bar{b}_2$ brings $\mathcal{B}$ to $\mathcal{E}$.

The rotation matrix is then

$$\underline{\underline{D}}_{3\text{-}1\text{-}2} = \begin{bmatrix} C_1C_3 - S_1S_2S_3 & -S_1C_2 & C_1S_3 + S_1S_2C_3 \\ S_1C_3 + C_1S_2S_3 & C_1C_2 & S_1S_3 - C_1S_2C_3 \\ -C_2S_3 & S_2 & C_2C_3 \end{bmatrix} \tag{33}$$

When requesting a rotation output by means of a sensor, it is necessary to perform the inverse operation: given a direction cosine matrix, find the corresponding Euler angles. The following process will yield the desired angles.

$$\phi_3 = \text{atan2}(-D_{31}, D_{33}); \quad \text{if} \quad D_{33} \neq 0; \tag{34}$$

$$\phi_2 = \text{atan2}(D_{32}, D_{33}C_3 - D_{31}S_3); \tag{35}$$

$$\phi_1 = \text{atan2}(-D_{12}, D_{22}). \tag{36}$$

It is clear that when $\theta = \pm\pi/2$, a singularity occurs.

3 Fixed frames

A fixed fame, $\mathcal{F} = [\mathbf{E}, \mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)]$, is defined as a triad with its origin at a fixed point in space, as depicted in fig. 6. Triad $\mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)$ is defined by three mutually orthogonal unit vectors, and the origin of the frame is defined by a point, denoted **E** on the figure.

A fixed fame can be defined in many different manner described in the following sections.

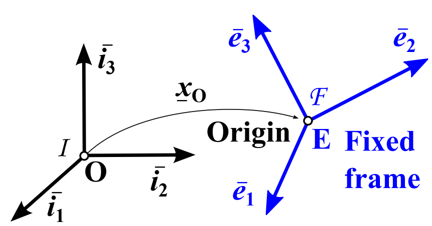

Figure 6: Configuration of a fixed frame.

3.1 One point and one triad

A cursory look at fig. 6 reveals that the most natural way of describing a fixed fame is to define its origin by a point, as discussed in section 1, and its orientation by a triad, as discussed in section 2. When defined in this manner, a fixed frame is a combination of the two concepts introduced in the two previous sections. Of course, the triad can be defined in many alternative manners, as discussed sections 2.1 to 2.8.

3.2 Two points and one vector

A fixed frame can be defined by two points and one vector. The two points are indicated in fig. 7 as points $\mathbf{P_1}$ and $\mathbf{P_2}$. The vector is selected as the third unit vector of a triad, denoted $\bar{n}_3$. In this case, it is convenient to define the triad by a single vector, as discussed in section 2.3. Point $\mathbf{P_1}$ defines the origin of the frame; points $\mathbf{P_1}$ and $\mathbf{P_2}$ define the orientation of unit vector $\bar{e}_1$ and plane $(\bar{e}_1, \bar{n}_3)$ coincides with plane $(\bar{e}_1, \bar{e}_2)$. A valid definition requires the satisfaction of the following two conditions: *(1)* $\|\underline{v}_1\| \neq 0$ and *(2)* $\tilde{v}_1\bar{n}_3 \neq \underline{0}$, where $\underline{v}_1 = \mathbf{P_2} - \mathbf{P_1}$.

The following algorithm is then used to define the fixed frame. *(1)* The origin of the fixed frame is selected to be point $\mathbf{P_1}$, *i.e.*, $\mathbf{E} = \mathbf{P_1}$. *(2)* Next, unit vector $\bar{e}_1$ is computed by normalization of vector $\underline{v}_1 = \mathbf{P_2} - \mathbf{P_1}$,

$$\underline{e}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}. \tag{37}$$

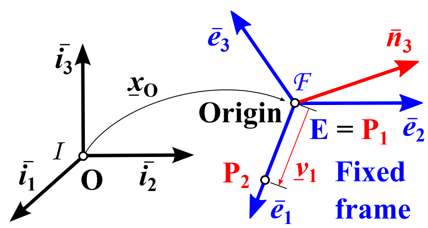

Figure 7: Definition of a fixed frame.

(3) Unit vector $\bar{e}_2$ is computed by orthogonalizing vector $\bar{n}_3$ against $\bar{e}_1$, then normalizing the result, to find

$$\underline{v}_2 = \bar{n}_3 - (\bar{e}_1^T\bar{n}_3)\bar{e}_1; \quad \bar{e}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|}. \tag{38}$$

(4) Finally, unit vector $\bar{e}_3$ is the vector product of unit vectors $\bar{e}_1$ and $\bar{e}_2$,

$$\bar{e}_3 = \tilde{e}_1\bar{e}_2. \tag{39}$$

3.3 Three points

Finally, a fixed frame can be defined by three points, which are indicated in fig. 8 as points $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$. Point $\mathbf{P_1}$ defines the origin of the frame; points $\mathbf{P_1}$ and $\mathbf{P_2}$ define the orientation of unit vector $\bar{e}_1$ and the plane defines by points $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$ coincides with plane $(\bar{e}_1, \bar{e}_2)$. A valid definition requires the satisfaction of the following condition: point $\mathbf{P_1}$, $\mathbf{P_2}$, and $\mathbf{P_3}$ cannot be collinear.

The following algorithm is then used to define the fixed frame. *(1)* The origin of the fixed frame is selected to be point $\mathbf{P_1}$, *i.e.*, $\mathbf{E} = \mathbf{P_1}$. *(2)* Next, unit vector $\bar{e}_1$ is computed by normalization of vector $\underline{v}_1 = \mathbf{P_2} - \mathbf{P_1}$,

$$\underline{e}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}. \tag{40}$$

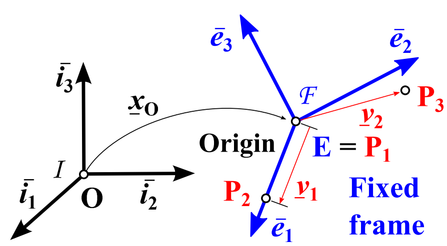

Figure 8: Definition of a fixed frame.

(3) Unit vector $\bar{e}_2$ is computed by orthogonalizing vector $\underline{v}_2 = \mathbf{P_3} - \mathbf{P_1}$ against $\bar{e}_1$, then normalizing the result, to find

$$\underline{v}_2' = \bar{v}_2 - (\bar{e}_1^T\bar{v}_2)\bar{e}_1; \quad \bar{e}_2 = \frac{\underline{v}_2'}{\|\underline{v}_2'\|}. \tag{41}$$

(4) Finally, unit vector $\bar{e}_3$ is the vector product of unit vectors $\bar{e}_1$ and $\bar{e}_2$,

$$\bar{e}_3 = \tilde{e}_1\bar{e}_2. \tag{42}$$

4 Definition of geometric entities with respect to a frame

The previous section have presented the definition of points, triads, and fixed frames with respect to the inertial frame, $\mathcal{F}_I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$. In many practical situations, geometric entities such as points, triads, or fixed frames are defined *with respect to a frame*, whose configuration with respect to the inertial frame is known, as illustrated in fig. 9.

Let frame $\mathcal{F}_E = [\mathbf{E}, \mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)]$ be a fixed frame defined by the inertial position of its origin, $\underline{x}_E$, and the orientation of its triad, $\underline{\underline{R}}_E$. Array $\underline{x}_E$ stores the components of the position vector of point $\mathbf{E}$ with respect to point $\mathbf{O}$, resolved in basis $\mathcal{I}$. Matrix $\underline{\underline{R}}_E$ stores the components of the rotation tensor that brings basis $\mathcal{I}$ to basis $\mathcal{E}$, resolved in basis $\mathcal{I}$. Alternatively, fixed frame $\mathcal{F}_E$ can also be defined by its motion tensor, $\underline{\underline{\mathcal{C}}}_E$.

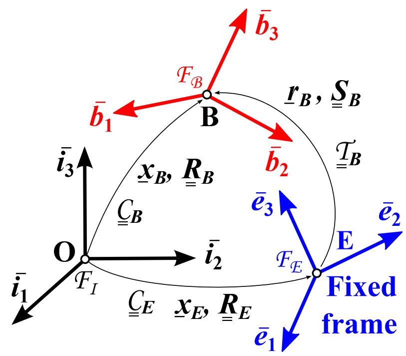

Figure 9: Point and triad defined with respect to a frame.

Figure 9 shows point $\mathbf{B}$ and triad $\mathcal{B}$, which define frame $\mathcal{F}_B = [\mathbf{B}, \mathcal{B} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)]$. Array $\underline{x}_B$ stores the components of the position vector of point $\mathbf{B}$ with respect to point $\mathbf{O}$, resolved in basis $\mathcal{I}$. Matrix $\underline{\underline{R}}_B$ stores the components of the rotation tensor that brings basis $\mathcal{I}$ to basis $\mathcal{B}$, resolved in basis $\mathcal{I}$. Alternatively, frame $\mathcal{F}_B$ can also be defined by its motion tensor, $\underline{\underline{\mathcal{C}}}_B$.

It is also possible to define point $\mathbf{B}$, triad $\mathcal{B}$, and frame $\mathcal{F}_B$ with respect to fixed frame $\mathcal{E}$. Array $\underline{r}_B$ stores the components of the position vector of point $\mathbf{B}$ with respect to point $\mathbf{E}$ and matrix $\underline{\underline{S}}_B$ stores the components of the rotation tensor that bring basis $\mathcal{E}$ to $\mathcal{B}$, both resolved in inertial basis $\mathcal{I}$. When resolved in fixed frame $\mathcal{F}_E$, the corresponding quantities are denoted $\underline{x}^*_B$ and $\underline{\underline{S}}^*_B$, respectively, where notation $(\cdot)^*$ indicates tensor components resolved in basis $\mathcal{E}$. Clearly, $\underline{r}^*_B = \underline{\underline{R}}^T_E \underline{r}_B$ and $\underline{\underline{S}}^*_B = \underline{\underline{R}}^T_E \underline{\underline{S}}_B \underline{\underline{R}}_E$. The motion tensor that brings frame $\mathcal{F}_E$ to $\mathcal{F}_B$ is denoted $\underline{\underline{\mathcal{T}}}_B$ and hence, $\underline{\underline{\mathcal{C}}}_B = \underline{\underline{\mathcal{T}}}_B \underline{\underline{\mathcal{C}}}_E$. When resolving the components of motion tensor $\underline{\underline{\mathcal{T}}}_B$ in frame $\mathcal{F}_E$, $\underline{\underline{\mathcal{C}}}_B = \underline{\underline{\mathcal{C}}}_E \underline{\underline{\mathcal{T}}}^*_B$.

Fixed frame $\mathcal{F}_E$ is defined in inertial frame $\mathcal{F}_\mathcal{I}$. Two situations are of practical interest.

1. Point $\mathbf{B}$, basis $\mathcal{B}$, or frame $\mathcal{F}_B$ are defined with respect to frame $\mathcal{F}_E$ and their definitions with respect to inertial frame $\mathcal{F}_\mathcal{I}$ are to be determined.

2. Point $\mathbf{B}$, basis $\mathcal{B}$, or frame $\mathcal{F}_B$ are defined with respect to inertial frame $\mathcal{F}_\mathcal{I}$ and their definitions with respect to fixed frame $\mathcal{F}_E$ are to be determined.

4.1 Inertial from relative definition

In this situation, point $\mathbf{B}$ and basis $\mathcal{B}$ are defined with respect to frame $\mathcal{F}_E$ by the components of the relative position of point $\mathbf{B}$ with respect to point $\mathbf{E}$, $\underline{r}^*_B$, and the components of the relative rotation tensor that brings basis $\mathcal{E}$ to $\mathcal{B}$, $\underline{\underline{S}}^*_B$, both resolved in basis $\mathcal{E}$. The components of the position vector of point $\mathbf{B}$ with respect to point $\mathbf{O}$, $\underline{x}_B$, and the components of the rotation tensor that bring basis $\mathcal{I}$ to $\mathcal{B}$, $\underline{\underline{R}}_B$, both resolved in basis $\mathcal{I}$ must be determined given the definition of fixed frame $\mathcal{F}_E$.

The position vector of point $\mathbf{B}$ with respect to the inertial frame is obtained from elementary vector algebra

$$\underline{x}_B = \underline{x}_E + \underline{r}_B = \underline{x}_E + \underline{\underline{R}}_E \underline{r}^*_B. \tag{43}$$

The orientation of basis $\mathcal{B}$ with respect to basis $\mathcal{I}$ is obtained easily

$$\underline{\underline{R}}_B = \underline{\underline{S}}_B \underline{\underline{R}}_E = \underline{\underline{R}}_E \underline{\underline{S}}^*_B. \tag{44}$$

The same relationships can be obtained more expeditiously from the motion tensors describing the configuration of the problem. Let Euler motion parameters $\hat{p}_E^T = \{\hat{q}_E^T, \hat{e}_E^T\}$, $\hat{p}_B^T = \{\hat{q}_B^T, \hat{e}_B^T\}$, and $\hat{p}_B^{*T} = \{\hat{q}_B^{*T}, \hat{e}_B^{*T}\}$ be associated with motion tensors $\underline{\underline{\mathcal{C}}}_E$, $\underline{\underline{\mathcal{C}}}_B$, and $\underline{\underline{\mathcal{T}}}_B^*$, respectively. Because $\underline{\underline{\mathcal{C}}}_B = \underline{\underline{\mathcal{C}}}_E \underline{\underline{\mathcal{T}}}_B^*$, the composition of motion formula implies

$$\hat{p}_B = \underline{\underline{\mathcal{A}}}(\hat{p}_E)\hat{p}_B^*, \tag{45}$$

which expands to

$$\begin{Bmatrix} \hat{q}_B \\ \hat{e}_B \end{Bmatrix} = \begin{bmatrix} \underline{\underline{A}}(\hat{e}_E) & \underline{\underline{A}}(\hat{q}_E) \\ \underline{\underline{0}} & \underline{\underline{A}}(\hat{e}_E) \end{bmatrix} \begin{Bmatrix} \hat{q}_B^* \\ \hat{e}_B^* \end{Bmatrix}. \tag{46}$$

Because $\hat{x}_E = 2\underline{\underline{B}}^T(\hat{e}_E)\hat{q}_E$ and $\hat{x}_B = 2\underline{\underline{B}}^T(\hat{e}_B)\hat{q}_B$, it follows that

$$\hat{x}_B = \underline{\underline{B}}^T(\hat{e}_E)\left[\underline{\underline{A}}(\hat{e}_E)\hat{r}_B^* + 2\hat{q}_E\right], \tag{47a}$$

$$\hat{e}_B = \underline{\underline{A}}(\hat{e}_E)\hat{e}_B^*, \tag{47b}$$

which echo eqs. (43) and (44). The displacement quaternions are $\hat{x}_B^T = \{0, \underline{x}_B^T\}$ and $\hat{r}_B^{*T} = \{0, \underline{r}_B^{*T}\}$.

The easiest way to obtain the desired result is to use composition of motion (45). Once bi-quaternion $\hat{p}_B$ are evaluated, other representations of the same result, such as displacement and rotation components, are obtained easily.

4.2 Relative from inertial definition

In this situation, point **B** and basis $\mathcal{B}$ are defined with respect to inertial frame $\mathcal{F}_\mathcal{I}$ by the components of the relative position of point **B** with respect to point **O**, $\underline{x}_B$, and the components of the relative rotation tensor that brings basis $\mathcal{I}$ to $\mathcal{B}$, $\underline{\underline{R}}_B$, both resolved in basis $\mathcal{I}$. The components of the position vector of point **B** with respect to point **E**, $\underline{r}_B^*$, and the components of the rotation tensor that bring basis $\mathcal{E}$ to $\mathcal{B}$, $\underline{\underline{S}}_B^*$, both resolved in basis $\mathcal{E}$ must be determined given the definition of fixed frame $\mathcal{F}_E$.

This problem is solved easily by reversing the process described in the previous section. The components of the relative position vector of point **B** with respect to point **E**, resolved in triad $\mathcal{E}$, follow from eq. (43)

$$\underline{r}_B^* = \underline{\underline{R}}_E^T(\underline{x}_B - \underline{x}_E). \tag{48}$$

Similarly, the components of the relative rotation tensor of basis $\mathcal{B}$ with respect to basis $\mathcal{E}$, resolved in triad $\mathcal{E}$, follow from eq. (44)

$$\underline{\underline{S}}_B^* = \underline{\underline{R}}_E^T \underline{\underline{R}}_B. \tag{49}$$

Because $\underline{\underline{\mathcal{T}}}_B^* = \underline{\underline{\mathcal{C}}}_E^{-1}\underline{\underline{\mathcal{C}}}_B$, the composition of motion formula implies $\hat{p}_B^* = \underline{\underline{\mathcal{A}}}^\dagger(\hat{p}_E)\hat{p}_B$ and expanding this formula yields the desired results

$$\hat{r}_B^* = \underline{\underline{A}}^T(\hat{e}_E)\left[\underline{\underline{B}}(\hat{e}_E)\hat{x}_B - 2\hat{q}_E\right], \tag{50a}$$

$$\hat{e}_B^* = \underline{\underline{A}}^T(\hat{e}_E)\hat{e}_B, \tag{50b}$$

which echo eqs. (48) and (49). Here again it is most expeditious to use the composition of motion formula.

5 Recursive definition of basic geometric entities

Many practical situations are more complex than those discussed thus far, because geometric entities might be defined recursively. For instance, a point might be defined with respect to a frame that itself is defined with respect to a second frame, which is defined with respect to the inertial frame. In such case, the transformation formulæ presented in section 4.1 must be applied recursively, until the configuration of all geometric entities have been resolved with respect to the inertial frame.

A further complicating factor comes from the fact that a frame could be defined by a combination of points and triads, which themselves could be defined with respect to potentially different frames. When the configuration of a geometric entity is known with respect to the inertial frame, the entity is said to be *resolved*, it is said to be *unresolved* in the opposite case. The following algorithm works recursively to resolve all geometric entities.

Flag all geometric entities as *unresolved*.
Loop over iterations

> **Loop over all geometric entities**
>
> > If the geometric entity is defined with respect to a *resolved frame*, find its absolute configuration using eqs. (43) and (44) and flag entity as *resolved*.
> >
> > else, skip geometric entity.
>
> **}**
>
> If some geometric entities are left *unresolved*, continue with the next iteration, else exit.

}

Of course, the success of the procedure is guaranteed only in the absence of circular references. This means that at each iteration, at least one unresolved geometric entity must be resolved; if not, the iteration process is stopped and an error message is printed.

6 Change of reference frame operations

Section 4 has focused on fixed frames, *i.e.*, frames that remain invariant in times. When dealing withe time dependant simulations, it is often convenient to present the results of the simulation as viewed by observers associated with different frames. Three cases can be identified.

1. The dynamic response of the system is viewed by the inertial observer: the results are "viewed by an inertial observer associated with frame $\mathcal{F}_{\mathcal{I}}$."

2. The dynamic response of the system is viewed by an inertial observer associated with a *fixed frame*, say frame $\mathcal{F}_E$: the results are "viewed by an inertial observer associated with frame $\mathcal{F}_E$." The formulæ presented in section 4 are applicable to this situation.

3. The dynamic response of the system is viewed by an observer associated with a *moving or reference frame*, say frame $\mathcal{F}_E$: the results are "viewed by an observer associated with moving or reference frame $\mathcal{F}_E$."

This section examines the third situation, depicted in fig. 10, where the moving or *reference frame* is denoted $\mathcal{F}_{0\mathcal{E}}$ and $\mathcal{F}_E$ at times $t = 0$ and t, respectively. In the reference and present configurations, at times $t = 0$ and t, respectively, the reference frame is defined as $\mathcal{F}_{E0} = [\mathbf{E}, \mathcal{E}_0 = (\bar{e}_{01}, \bar{e}_{02}, \bar{e}_{03})]$ and $\mathcal{F}_E = [\mathbf{E}, \mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)]$, respectively. At the same times, a geometric entity of the system is defined by frames $\mathcal{F}_{B0} = [\mathbf{B}, \mathcal{B}_0 = (\bar{b}_{01}, \bar{b}_{02}, \bar{b}_{03})]$ and $\mathcal{F}_B = [\mathbf{B}, \mathcal{B} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)]$, respectively.

6.1 Configurations of the frames

In the reference configuration, frames $\mathcal{F}_{B0}$ and $\mathcal{F}_{E0}$ are defined by the position vectors of their origins, denoted $\underline{x}_{B0}$ and $\underline{x}_{E0}$, respectively, and the rotation tensors of their bases, denoted $\underline{\underline{R}}_{B0}$ and $\underline{\underline{R}}_{E0}$, respectively; an alternative representation is provided by motion tensors $\underline{\underline{\mathcal{C}}}_{B0}$ and $\underline{\underline{\mathcal{C}}}_{E0}$, respectively. In the present configuration, frames $\mathcal{F}_B$ and $\mathcal{F}_E$ are defined by the displacement vectors of their origins, denoted $\underline{u}_B$ and $\underline{u}_E$, respectively, and the rotation tensors of their bases, denoted $\underline{\underline{R}}_B$ and $\underline{\underline{R}}_E$, respectively; an alternative representation is provided by motion tensors $\underline{\underline{\mathcal{C}}}_B$ and $\underline{\underline{\mathcal{C}}}_E$, respectively. All quantities are resolved in inertial frame $\mathcal{F}_\mathcal{I}$ and correspond the the response of the system as "viewed by an inertial observer associated with frame $\mathcal{F}_\mathcal{I}$."

In the reference configuration, the relative configuration of frame $\mathcal{F}_{B0}$ with respect to frame $\mathcal{F}_{E0}$ is defined by the relative position vectors of their origins, denoted $\underline{r}_{B0}$, and the relative rotation tensor of their bases, denoted $\underline{\underline{S}}_{B0}$; an alternative representation is provided by the relative motion tensor $\underline{\underline{\mathcal{T}}}_{B0}$. Finally, in the present configuration, the corresponding quantities are denoted $\underline{r}_B$ and $\underline{\underline{S}}_B$, respectively, and the relative motion tensor is denoted $\underline{\underline{\mathcal{T}}}_B$.

When resolved in basis $\mathcal{E}_0$, the components of the relative position vector and relative rotation tensor are denoted $\underline{r}^\natural_{B0} = \underline{\underline{R}}^T_{E0}\underline{r}_{B0}$ and $\underline{\underline{S}}^\natural_{B0} = \underline{\underline{R}}^T_{E0}\underline{\underline{S}}_{B0}\underline{\underline{R}}_{E0}$, respectively. These quantities provide the configuration of frame $\mathcal{F}_{B0}$ as "viewed by an observer associated with frame $\mathcal{F}_{E0}$."

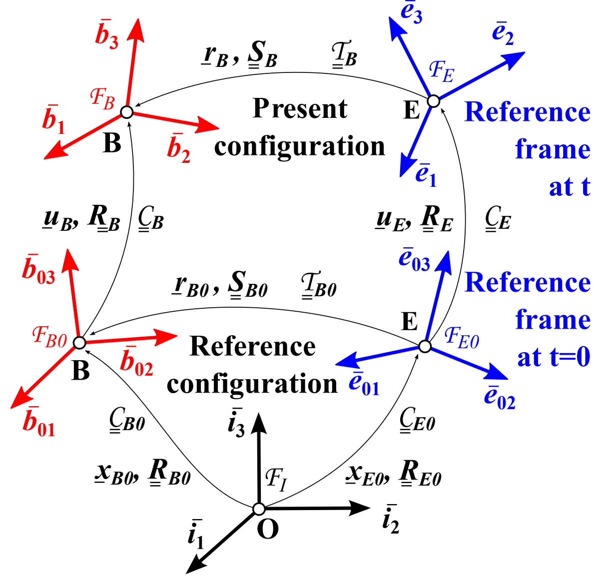

Figure 10: Change of reference frame

Similarly, when resolved in basis $\mathcal{E}$, the components of the relative position vector and relative rotation tensor are denoted $\underline{r}^\natural_B = \underline{\underline{R}}^T_E\underline{r}_B$ and $\underline{\underline{S}}^\natural_B = \underline{\underline{R}}^T_E\underline{\underline{S}}_B\underline{\underline{R}}_E$, respectively. These quantities provide the configuration of frame $\mathcal{F}_B$ as "viewed by an observer associated with frame $\mathcal{F}_E$." This section relates the quantities viewed by the inertial observer to those viewed by the observer associated with reference frame $\mathcal{F}_E$.

6.2 Configuration viewed by a moving observer

In the reference configuration, the relationships derived in section 4.2 can be used directly, leading to

$$\underline{r}^\natural_{B0} = \underline{\underline{R}}^T_{E0}(\underline{x}_{B0} - \underline{x}_{E0}), \tag{51a}$$
$$\underline{\underline{S}}^\natural_{B0} = \underline{\underline{R}}^T_{E0}\underline{\underline{R}}_{B0}, \tag{51b}$$

that mirror eqs. (48) and (49).

Let the Euler motion parameters associated motion tensors $\underline{\underline{\mathcal{C}}}_{E0}$, $\underline{\underline{\mathcal{C}}}_{B0}$, and $\underline{\underline{\mathcal{T}}}^\natural_{B0}$ be denoted $\hat{p}^T_{E0} = \{\hat{q}^T_{E0}, \hat{e}^T_{E0}\}$, $\hat{p}^T_{B0} = \{\hat{q}^T_{B0}, \hat{e}^T_{B0}\}$, and $\hat{p}^{\natural T}_{B0} = \{\hat{q}^{\natural T}_{B0}, \hat{e}^{\natural T}_{B0}\}$, respectively. Because $\underline{\underline{\mathcal{T}}}^\natural_{B0} = \underline{\underline{\mathcal{C}}}^{-1}_{E0}\underline{\underline{\mathcal{C}}}_{B0}$, composition of motion yields $\hat{p}^\natural_{B0} = \underline{\underline{\mathcal{A}}}^\dagger(\hat{p}_{E0})\hat{p}_{B0}$ and expanding this matrix relationship yields expressions equivalent to those of eqs. (51a) and (51b).

In the present configuration, the following relationships are obtained easily,

$$\underline{r}_B^\natural = (\underline{\underline{R}}_E \underline{\underline{R}}_{E0})^T [(\underline{x}_{B0} + \underline{u}_B) - (\underline{x}_{E0} + \underline{u}_E)], \tag{52a}$$

$$\underline{\underline{S}}_B^\natural = (\underline{\underline{R}}_E \underline{\underline{R}}_{E0})^T (\underline{\underline{R}}_B \underline{\underline{R}}_{B0}). \tag{52b}$$

Let the Euler motion parameters associated motion tensors $(\underline{\underline{\mathcal{C}}}_E \underline{\underline{\mathcal{C}}}_{E0})$, $(\underline{\underline{\mathcal{C}}}_B \underline{\underline{\mathcal{C}}}_{B0})$, and $\underline{\underline{\mathcal{T}}}_B^\natural$ be denoted $\hat{p}_E^T = \{\hat{q}_E^T, \hat{e}_E^T\}$, $\hat{p}_B^T = \{\hat{q}_B^T, \hat{e}_B^T\}$, and $\hat{p}_B^{\natural T} = \{\hat{q}_B^{\natural T}, \hat{e}_B^{\natural T}\}$, respectively. Because $\underline{\underline{\mathcal{T}}}_B^\natural = (\underline{\underline{\mathcal{C}}}_E \underline{\underline{\mathcal{C}}}_{E0})^{-1} (\underline{\underline{\mathcal{C}}}_B \underline{\underline{\mathcal{C}}}_{B0})$, the composition of motion formula yields $\hat{p}_B^\natural = \underline{\underline{\mathcal{A}}}^\dagger(\hat{p}_E)\hat{p}_B$, which can be expanded to yield expression equivalent to eqs. (52a) and (52b).

6.3 Change of configuration viewed by a moving observer

The position vector of point **B** with respect to point **E** in the reference configuration is $\underline{r}_{B0} = \underline{x}_{B0} - \underline{x}_{E0}$ and the components of this vector resolved in basis $\mathcal{E}_0$ are $\underline{r}_{B0}^\natural = \underline{\underline{R}}_{E0}^T(\underline{x}_{B0} - \underline{x}_{E0})$, see eq. (51a). If frame $\mathcal{F}_B$ were to remain rigidly connected to frame $\mathcal{F}_E$, their relative configuration would remain unchanged and hence, at any time, $\underline{r}_B^\natural = \underline{r}_{B0}^\natural = \underline{\underline{R}}_{E0}^T(\underline{x}_{B0} - \underline{x}_{E0})$.

The components of *displacement vector* of point **B** with respect to point **E** as viewed by an observer on frame $\mathcal{F}_E$ in the final configuration are

$$\underline{u}_B^\natural = (\underline{\underline{R}}_E \underline{\underline{R}}_{E0})^T [(\underline{x}_{B0} + \underline{u}_B) - (\underline{x}_{E0} + \underline{u}_E)] - \underline{\underline{R}}_{E0}^T(\underline{x}_{B0} - \underline{x}_{E0}). \tag{53}$$

The first term is the position of point **B** with respect to point **E** as given by eq. (52a) and the second term gives the relative position of the same point if it were rigidly connected to frame $\mathcal{F}_E$; the difference provides the displacement of point **B** with respect to point **E** as viewed by an observer on frame $\mathcal{F}_E$, resolved in the same frame.

Similarly, eq. (51b) gives the relative rotation of basis $\mathcal{B}_0$ with respect to basis $\mathcal{E}_0$ as $\underline{\underline{S}}_{B0}^\natural = \underline{\underline{R}}_{E0}^T \underline{\underline{R}}_{B0}$, when resolved in basis $\mathcal{E}_0$. Here again, if frame $\mathcal{F}_B$ were to remain rigidly connected to frame $\mathcal{F}_E$, their relative configuration would remain unchanged and hence, at any time, $\underline{\underline{S}}_B^\natural = \underline{\underline{S}}_{B0}^\natural = \underline{\underline{R}}_{E0}^T \underline{\underline{R}}_{B0}$.

Let rotation tensor $\underline{\underline{Q}}_B$ be the *change in orientation* of basis $\mathcal{B}$ with respect to basis $\mathcal{E}$, between the reference and final configurations. The components of this change in orientation of basis $\mathcal{B}$ as viewed by an observer on frame $\mathcal{F}_E$ and resolved in the same frame are

$$\underline{\underline{Q}}_B^\natural = \underline{\underline{S}}_B^\natural \underline{\underline{S}}_{B0}^{\natural T} = \underline{\underline{R}}_{E0}^T (\underline{\underline{R}}_E^T \underline{\underline{R}}_B) \underline{\underline{R}}_{E0}. \tag{54}$$

Here again, these results are obtained more expeditiously from the composition of motion formula. Because $\underline{\underline{\mathcal{T}}}_{B0}^\natural = \underline{\underline{\mathcal{C}}}_{E0}^{-1} \underline{\underline{\mathcal{C}}}_{B0}$ and $\underline{\underline{\mathcal{T}}}_B^\natural = (\underline{\underline{\mathcal{C}}}_E \underline{\underline{\mathcal{C}}}_{E0})^{-1} (\underline{\underline{\mathcal{C}}}_B \underline{\underline{\mathcal{C}}}_{B0})$, the change of motion tensor is $\underline{\underline{\mathcal{Q}}}_B^\natural = \underline{\underline{\mathcal{T}}}_B^\natural \underline{\underline{\mathcal{T}}}_{B0}^{\natural -1}$. Because the Euler motion parameters of motion tensors $\underline{\underline{\mathcal{T}}}_{B0}^\natural$ and $\underline{\underline{\mathcal{T}}}_B^\natural$ are $\hat{p}_{B0}^\natural$ and $\hat{p}_B^\natural$, respectively, the Euler motion parameters of the change of motion tensor is $\underline{\underline{\mathcal{Q}}}_B^\natural$, denoted $\hat{q}^\natural$, are given by the composition of motion formula as $\hat{q}^\natural = \underline{\underline{\mathcal{B}}}^\dagger(\hat{p}_{B0}^\natural)\hat{p}_B^\natural$.

6.4 Velocity components viewed by a moving observer

Not sure this is right???? Does not seem to be used in Dymore????

Let the final configuration of the system be time-dependent. The inertial velocities of points **A** and **B**, denoted $\underline{v}^A$ and $\underline{v}^B$, respectively, are easily found as $\underline{v}^A = \underline{\dot{u}}^A$ and $\underline{v}^B = \underline{\dot{u}}^B$, respectively. Similarly, the angular velocity vectors of bases $\mathcal{E}^A$ and $\mathcal{E}^B$, denoted $\underline{\omega}^A$ and $\underline{\omega}^B$, respectively, are easily found as $\underline{\omega}^A = \text{axial}(\underline{\underline{\dot{R}}}^A \underline{\underline{R}}^{AT})$ and $\underline{\omega}^B = \text{axial}(\underline{\underline{\dot{R}}}^B \underline{\underline{R}}^{BT})$, respectively. All these vectors are expressed by their components in the inertial frame $\mathcal{I}$; *i.e.*, all quantities are "viewed by an inertial observer."

The components of the velocity vector of point **B** as viewed by an observer on frame $\mathcal{F}^A$, denoted $\underline{v}^{*B/A}$, are readily found by taking a time derivative of eq. (53) to find

$$\underline{v}^{*B/A} = \left(\underline{\underline{R}}^A \underline{\underline{R}}_0^A\right)^T \left\{\tilde{\omega}^{AT}\left[\left(\underline{u}_0^B + \underline{u}^B\right) - \left(\underline{u}_0^A + \underline{u}^A\right)\right] + \left(\underline{v}^B - \underline{v}^A\right)\right\}. \quad (55)$$

Similarly, the components of the angular velocity vector of basis $\mathcal{E}^B$ as viewed by an observer on frame $\mathcal{F}^A$ are $\underline{\omega}^{*B/A} = \text{axial}(\dot{\underline{\underline{Q}}}^{*B/A}\underline{\underline{Q}}^{*B/AT})$, and it the follows from eq. (54) that

$$\underline{\omega}^{*B/A} = \left(\underline{\underline{R}}^A \underline{\underline{R}}_0^A\right)^T \left(\underline{\omega}^B - \underline{\omega}^A\right). \quad (56)$$

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[1] O.A. Bauchau. *Flexible Multibody Dynamics*. Springer, Dordrecht, Heidelberg, London, New-York, 2011.